Math 83: Midterm 2 practice exam and answers

For problems 1-3 describe the maximum interval of existence to the given I.V.P.

1. $xy'' - x^2y' + xy = xy^2$, $y(1) = 1$, $y'(1) = 0$.

   A: Equation is nonlinear because of term y^2 (right-hand-side). Only local existence in a neighbourhood of $t = 1$ can be assured.

2. $ty'' - t^2y' + (t + t^2)y = t\cos(t)$, $y(1) = 1$, $y'(1) = 0$.

   A: Division by t ($t \neq 0$) is possible. Coefficients are continuous for all times, maximum interval (in forward time) is $t \in [1, \infty)$.

3. $(t - 2)y'' - t^2y' + (t + t^2)y = \cos(t)$, $y(1) = 1$, $y'(1) = 0$.

   A: Division by $t - 2$ brings forth the problem at $t = 2$ (denominator in all coefficients) . Maximum interval (again, in forward time) is therefore $t \in [1, 2)$.

For problems 4 and 5 compute the Wronskian determinant of two independent solutions to the given ODE.

4. $ty'' - 2y' + \left(t + \frac{2}{t}\right) y = 0.$

   A: Coefficient $p(t)$ is $p = -2/t$. Abel's theorem shows $W(t) = \text{const.}t^2$.

5. $ty'' - 2y' + \left(-t + \frac{2}{t}\right) y = 0.$

   A: Same p, so again $W(t) = \text{const.}t^2$.

6. Verify that a solution of Problem 4 is $y_1(t) = t\cos t$ and use the method of reduction of order to write a first order DE for a second independent solution. Do not solve this equation, just indicate explicitly the coefficient $p(t)$ and the right-hand side function $g(t)$.

   A: First option: set $y_2(t) = v(t)y_1(t)$ get equation for $v(t)$,

$$v'' + 2v' \left(\frac{y_1'}{y_1} - \frac{1}{t}\right) = 0.$$

   This equation is first order for auxiliary variable $r = v'$, with

$$p(t) = 2\left(\frac{y_1'}{y_1} - \frac{1}{t}\right) = -2\tan t,$$

   and $g(t) = 0$.

Second option (quicker): from definition of Wronskian determinant and from previous calculation, get, upon dividing by y_1,

$$y_2' - y_2 \left(\frac{y_1'}{y_1} \right) = \text{const.} \left(\frac{t^2}{y_1} \right)$$

and substitute expression for y_1 to get

$$p(t) = \frac{1}{t}(t \tan t - 1),$$

and

$$g(t) = \text{const.} \frac{t}{\cos t}.$$

Notice the differences (and similarities) with the equation from first option. For the record, solving by either option for y_2, yields (perhaps not unexpectedly) $y_2(t) = t \sin t$.

7. Verify that a solution of Problem 5 is $y_1(t) = te^t$ and use the method of reduction of order to write a first order DE for a second independent solution. Do not solve this equation, just indicate explicitly the coefficient $p(t)$ and the right-hand side function $g(t)$.

A: Same as before,

$$v'' + 2v' \left(\frac{y_1'}{y_1} - \frac{1}{t} \right) = 0,$$

first order for auxiliary variable $r = v'$, with

$$p(t) = 2 \left(\frac{y_1'}{y_1} - \frac{1}{t} \right) = 2,$$

and $g(t) = 0$. Or,

$$y_2' - y_2 \left(\frac{y_1'}{y_1} \right) = \text{const.} \left(\frac{t^2}{y_1} \right)$$

with

$$p(t) = -\frac{1+t}{t},$$

and

$$g(t) = \text{const.} te^{-t}.$$

Again notice the differences (and similarities) with the first option, and again, for the record and expectedly $y_2(t) = te^{-t}$ by either option.

Problems 8 and 9 solved in detail in class.

10. Are the two functions $y_1 = e^{2t}$, $y_2 = e^{-2t}$ linearly independent?

A: Wronskian determinant is $W(t) = -4 \neq 0$, so linearly independent.