$(\lambda+2)^2 - 9 = 0 \qquad \lambda = -2 \pm 3 = \begin{cases} -5 \\ 1 \end{cases}$

Math 83: Final Exam Practice Problems

The following are practice problems for the final exam. For each differential equation, classify it (e.g. its order, linear or nonlinear, homogeneous or nonhomogeneous, a system, etc.). This should help you decide how to solve it.

1. Solve the differential equation subject to the given initial condition.

$$e^y \frac{dy}{dx} = e^x + x, \qquad y(0) = 0.$$

$e^y = e^x + \frac{x^2}{2} + C$

$y = \log\left(e^x + \frac{x^2}{2}\right)$

2. Find an expression for the solution of

$$\frac{dy}{dx} - x^3 y = (1 - x^3)e^x.$$

hom: $\frac{1}{y}\frac{dy}{dx} = x^3 \qquad \log y_h = \frac{x^4}{4} \qquad y_h = e^{x^4/4}$

$y_p = e^{x^4/4}\int^x ds\,(1-x^3)e^x e^{-\frac{x^4}{4}}$

3. Solve

$$4y'' + 8y' = e^{-2x} + x,$$

using your favorite method.

$y' = r \qquad 4r' + 8r = e^{-2x} + x$

or $\quad 4y' + 8y = -\frac{1}{2}e^{-2x} + \frac{x^2}{2}$

$y' + 2y = -\frac{1}{8}e^{-2x} + \frac{x^2}{8}$

4. Consider the following linear system.

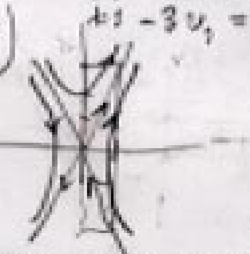
$$x_1' = -2x_1 + 9x_2,$$
$$x_2' = x_1 - 2x_2.$$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -2 & 9 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$(\lambda+2)^2 = 9 \Rightarrow \lambda = \pm 3 - 2 = \begin{Bmatrix} -5 \\ 1 \end{Bmatrix}$

(a) Write this system in matrix form. (b) Find its general solution. (c) Plot the general solution in the phase-plane $(x_1\ x_2)$.

$\lambda = -5: \ 3v_1 = -9v_2 \quad \begin{pmatrix} -3 \\ 1 \end{pmatrix} \qquad \lambda = 1: \ -3v_1 = -9v_2 \quad \begin{pmatrix} 3 \\ 1 \end{pmatrix}$



5. Consider the linear system $\vec{x}' = A\vec{x}$, where

$$A = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}.$$

$(\lambda+1)^2 = 1 \quad \lambda = -1 \pm i$

$(-1+i)v_1 - v_2 = 0 \qquad v_1 = -i v_2 \qquad \begin{pmatrix} -i \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix}$

Find the general solution to this system. Express your answer in terms of real functions.

$\begin{pmatrix} -i \\ 1 \end{pmatrix} e^{-t}(\cos t + i\sin t) = e^t\begin{pmatrix} -i\cos t + \sin t \\ \cos t + i\sin t \end{pmatrix} = e^t\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, e^t\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$

6. Given the mass–spring equation

$$mu'' + \gamma u' + ku = 0,$$

where m, γ and k are given constants, you should be able to

(a) Write this problem as a system of two first-order differential equations and find the general solution using the eigenvalue and eigenvector approach.

$u'' + \frac{\gamma}{m}u' + \frac{k}{m}u = 0 \quad 1 \quad u'' + \alpha u' + \omega_0^2 u = 0$

$\begin{pmatrix} u \\ u' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -\alpha \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix} \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -\omega_0^2 & -\alpha-\lambda \end{vmatrix} = +\lambda\alpha + \lambda^2 + \omega_0^2 = 0$

$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\omega_0^2}}{2}$