numerical solution of PDE's but keep in mind that the same methods are widely applicable.

3. Typical problems involving ODE's and PDE's

Now that we have arrived at the general form of PDE's which are of interest in many applications we can turn to actually finding solutions. An important first observation is that specifying the equation to be solved does not allow a unique solution. We must also specify additional *boundary* and/or *initial conditions.* Just as we have important special equations such as the advection or the Laplace equation, there are a number of important, archetypal problems involving ODE's and PDE's.

3.1. Initial value problem for ODE's. The simplest problem is the *initial value problem* for a first order system of ODE's

$$\left\{ \begin{array}{c} \mathbf{q}' = \mathbf{f}(t,\mathbf{q}) \\ \mathbf{q}(t=0) = \mathbf{q}_0 \end{array} \right. . \tag{3.1}$$

This also encompasses initial value problems for ODE's of higher order since an ODE of order p can always be rewritten as a system of p ODE's of order 1. To exemplify, consider

$$q''' = g(t,q,q',q'') . \tag{3.2}$$

By introducing the auxilliary functions

$$r = q', \ s = q'' \tag{3.3}$$

we obtain the system

$$\frac{d}{dt}\left[\begin{array}{c} q \\ r \\ s \end{array} \right] = \left[\begin{array}{c} r \\ s \\ g(t,q,r,s) \end{array} \right] \tag{3.4}$$

which is of the form (3.1).

3.2. Boundary value problem for ODE's. For ODE's of order 2 or greater or for systems of two or more ODE's one can meaningfully impose boundary conditions at distinct points within the computational domain. The archetypal ODE boundary value problem is for a second order ODE with conditions at the end points of the computational domain

$$\left\{ \begin{array}{c} q'' = f(t,q,q') \\ q(a) = q_1 \\ q(b) = q_2 \end{array} \right. . \tag{3.5}$$

Instead of the function values, its derivatives might be specified such as in

$$\left\{ \begin{array}{c} q'' = f(t,q,q') \\ q'(a) = r_1 \\ q'(b) = r_2 \end{array} \right. . \tag{3.6}$$

3.3. Initial value problems for PDE's. We can pose boundary and/or initial value conditions for PDE's also. It should be noted that not all combinations of PDE's and boundary conditions are compatible. For a large class of phenomena modeled by differential equations we have a reasonable expectation that small changes in the boundary conditions should lead to small changes in the solution. We would also expect the solution to exist and be unique. This means that the solution should depend continuously on the boundary data. Problems for which this holds are said to be *well posed in the sense of Hadamard* and we shall concentrate almost exclusively on this type of problems. Note that not all phenomena modeled need to behave this way as shown by the sensitive dependence on initial data shown in chaotic behavior.

The PDE initial value problem (IVP) most closely related to (3.1) is that posed for the scalar advection equation

$$\begin{cases} \dfrac{\partial q}{\partial t} + u\dfrac{\partial q}{\partial x} = \sigma(x,t,q) \\ q(x,t=0) = q_0(x), \ -\infty < x < \infty \end{cases} .$$

This problem is well posed and straight forward to solve as we shall see later on. The advection equation is the simplest example of the class *hyperbolic* PDE's. The name is a result of historical accident; the first time PDE's were actively studied scientists were interested in second order PDE's the classification of which can be related to that of quadratic curves.

We can also consider PDE's of similar form for vector variables

$$\begin{cases} \dfrac{\partial \mathbf{q}}{\partial t} + \mathbf{A}\dfrac{\partial \mathbf{q}}{\partial x} = \sigma(x,t,\mathbf{q}) \\ \mathbf{q}(x,t=0) = \mathbf{q}_0(x), \ -\infty < x < \infty \end{cases} . \tag{3.7}$$

This IVP is well posed if the eigenvectors of the matrix $\mathbf{A}$ form a complete set.

3.4. Boundary value problems for PDE's. The archetypal boundary value problems are posed for the Poisson equation. Here are the most commonly encountered problems exemplified for the 2D Poisson equation.

(1) *Dirichlet problem*, in which the values of the unknown function are given along the solution domain's boundary

$$\begin{cases} \dfrac{\partial^2 q}{\partial x^2} + \dfrac{\partial^2 q}{\partial y^2} = \sigma(x,y,q), \ (x,y) \in \Omega \\ q(x,y) = F(x,y), \ (x,y) \in \partial\Omega \end{cases} . \tag{3.8}$$

Here, and in the following, Ω is the domain over which the problem is defined and $\partial\Omega$ is its boundary.

(2) *Neumann problem*, in which the values of the normal derivative of the unknown function are given along the solution domain's boundary

$$\begin{cases} \dfrac{\partial^2 q}{\partial x^2} + \dfrac{\partial^2 q}{\partial y^2} = \sigma(x,y,q), \ (x,y) \in \Omega \\ \dfrac{\partial q}{\partial n}(x,y) = F(x,y), \ (x,y) \in \partial\Omega \end{cases} . \tag{3.9}$$

(3) *Robin problem*, in which a linear combination of the function and its normal derivative are given on the boundary

$$\left\{ \begin{array}{c} \dfrac{\partial^2 q}{\partial x^2} + \dfrac{\partial^2 q}{\partial y^2} = \sigma(x, y, q),\ (x, y) \in \Omega \\ q(x, y) + k(x, y)\dfrac{\partial q}{\partial n}(x, y) = F(x, y),\ (x, y) \in \partial\Omega \end{array} \right. . \tag{3.10}$$

3.5. Mixed-type problems for PDE's. A number of PDE's require both initial and boundary value conditions. The typical case is given by the problem of solving the heat equation on a finite strip $a \leq x \leq b$

$$\left\{ \begin{array}{c} \dfrac{\partial q}{\partial t} = \alpha \dfrac{\partial^2 q}{\partial x^2} + \sigma(x, t, q), \\ q(a, t) = F_1(t),\ q(b, t) = F_2(t) \\ q(x, t = 0) = q_0(x) \end{array} \right. \tag{3.11}$$