Leapfrog method.

$$Q^{n+2} = Q^n + 2k\lambda Q^{n+1} \tag{6.61}$$

$$\rho(\zeta) = \zeta^2 - 1,\ \sigma(\zeta) = 2\zeta \tag{6.62}$$

The roots of $\rho(\zeta)$ are $\zeta_1 = 1$, $\zeta_2 = -1$ so the method is zero-stable. The absolute stability region is defined by

$$\left| z \pm \sqrt{z^2+1} \right| \leq 1 \tag{6.63}$$

6.3.4. *Boundary locus method.* Regions of absolute stability defined by conditions such as (6.60) or (6.63) are difficult to analyze. A useful tool to simplify the analysis is the *boundary locus method.* We are interested in regions of the values of z for which the roots of $\pi(\zeta; z)$ satisfy the stability condition. In general z can take complex values. As z varies a particular root of π takes different values; the interesting event is when the absolute value of a root is unity. We can therefore consider the problem of determining the curve in the complex z-plane where one of the roots of $\pi(\zeta; z)$ has absolute value unity. If $|\zeta| = 1$ then we can write $\zeta = e^{i\theta}$ and

$$\pi(e^{i\theta}; z) = \rho(e^{i\theta}) - z\sigma(e^{i\theta}) = 0 \tag{6.64}$$

from whence

$$z(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}\ . \tag{6.65}$$

The curve defined by (6.65) is known as the *boundary locus.* In the boundary locus method we trace $z(\theta)$. This delimits the complex z-plane into regions. We then take an arbitrary point z^* within each region and find the roots of $\pi(\zeta; z^*) = 0$. If the roots satisfy the stability condition then that region of the complex z-plane is a region of absolute stability; if not it is a region of absolute instability.

Example 4. *For the trapezoidal method we have*

$$z(\theta) = 2\frac{e^{i\theta}-1}{e^{i\theta}+1} = 2i\tan\frac{\theta}{2} \tag{6.66}$$

so the boundary locus is the imaginary axis. We have two regions. To the left of the imaginary axis the root of $\pi(\zeta; z = -1)$ *is* $\zeta_1 = 1/3$ *which satisfies the stability condition. To the right the root of* $\pi(\zeta; 1)$ *is* $\zeta_1 = 3$ *which does not satisfy the stability condition. We conclude that* $\operatorname{Im} z \leq 0$ *is the region of absolute stability for the trapezoidal method.*

Example 5. *For the leapfrog method we have*

$$z(\theta) = \frac{e^{2i\theta}-1}{e^{i\theta}} = 2i\sin\theta \tag{6.67}$$

so the boudary locus is the segment from $-i$ *to* i *along the imaginary axis. There are two regions. One is outside of this segment. The roots of* $\pi(\zeta; z = 1)$ *are* $\zeta_{1,2} = 1 \pm \sqrt{2}$ *that do not satisfy the stability condition. The other region is the* $-i$ *to* i *segment. The roots of* $\pi(\zeta; z = 0)$ *are* $\zeta_{1,2} = \pm 1$ *which satisfy the stability condition. The region of absolute stability for the leapfrog method is the segment from* $-i$ *to* i *along the imaginary axis.*

6.3.5. *Conditions for convergence.* With the above preparation we are able to state the main result of the theory.

THEOREM 3. *A LMM is convergent is it is stable and consistent.*

Note that the general characterization “stable” is used in the above theorem. We can use any of the definitions of stability introduced above; we obtain different types of convergence. When “zero-stability” is used we obtain that the LMM converges to the solution of the IVP at a fixed time T as $k \to 0$ but $nk = T$. When “absolute stability” is used we obtain asymptotic convergence as $n \to \infty$.