CHAPTER 4

Fourier Analysis of Common Linear Partial Differential Equations

1. Fourier Series

Fourier transform techniques are useful in the study of PDE's in many ways. Linear equations can often be directly solved by Fourier transforms. Such solutions are useful in their own right and also as test cases for validation of numerical algorithms intended for more complicated, non-linear problems. Thinking about the properties of an analytical or numerical solution both in real space and in Fourier space brings many insights that guide algorithmic development and analysis. Here we'll briefly go over the basic results from Fourier analysis that are especially useful in numerical work. Most results shall be presented in brief. Further detail will be presented when spectral methods are studied later.

Consider a real function f defined on $[-L, L]$. The function may be represented by a trigonometric or Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos \xi_k x + b_k \sin \xi_k x) \tag{1.1}$$

with $\xi_k = k\pi/L$. Note that the representation (1.1) implicitly prolongs f by periodicity outside of the original definition interval $[-L, L]$. Multiplying (1.1) by $\cos \xi_k x$, $\sin \xi_k x$ and integrating the result over $[-L, L]$ gives the Fourier coefficients

$$a_k = \frac{1}{L} \int_{-L}^{L} f(y) \cos \xi_k \; y \; dx, \; b_k = \frac{1}{L} \int_{-L}^{L} f(y) \sin \xi_k \; y \; dy \; . \tag{1.2}$$

Replacing the coefficients (1.2) back into (1.1) leads to

$$f(x) = \frac{1}{2L} \int_{-L}^{L} f(y) \; dy + \sum_{k=1}^{\infty} \frac{1}{L} \int_{-L}^{L} f(y) \cos \xi_k (y - x) dy \tag{1.3}$$

from which the Fourier transform can be defined in the limit $L \to \infty$.

Fourier series are useful in determining solutions to linear PDE's in conjunction with separation of variables.

Example 6. *Determine the temperature distribution in a bar given the initial temperatures $f(x)$ and the endpoint temperatures at all times t, $g_0(t)$, $g_1(t)$. The problem is modeled by the heat equation with initial and boundary conditions*

$$\begin{cases} q_t = q_{xx} \\ q(x, t=0) = f(x) \\ q(x=0, t) = g_0(t), \; q(x=1, t) = g_1(t) \end{cases} . \tag{1.4}$$

Physical units have been chosen so that the bar length and diffusion coefficient are equal to one. We can solve the problem by separation of variables by assuming that

$$q(x,t) = X(x)T(t) \tag{1.5}$$

which leads to

$$\frac{T'}{T} = \frac{X''}{X} \ . \tag{1.6}$$

The lhs depends only on t, the rhs only x. The relation has to be true for all x, t which are independent variables so the only possiblity is that both T'/T and X''/X are constant.

$$\frac{T'}{T} = \frac{X''}{X} = C \ . \tag{1.7}$$

Solving for T leads to $T(t) = T_0 e^{Ct}$. Physical reasoning leads to $C < 0$ since the temperature in the bar cannot increase without bound. Let $C = -a^2$. Solving for X then leads to $X(x) = A_a \cos ax + B_a \sin ax$. Note that the heat equation is satisfied for any a. Since the equation is linear, any linear combination of solutions is also a solution, so the most general form of the solution is

$$q(x,t) = \sum_a \left(A'_a \cos ax + B'_a \sin ax\right) e^{-a^2 t} \tag{1.8}$$

with $A'_a = T_0 A_a$, $B'_a = T_0 B_a$ and the sum being taken over all possible values of a. The specific values of the coefficients shall be determined by the initial and boundary conditions

$$q(x,0) = f(x) = \sum_a \left(A'_a \cos ax + B'_a \sin ax\right) \tag{1.9}$$

$$q(0,t) = g_0(t) = \sum_a A'_a e^{-a^2 t}, \ q(1,t) = g_1(t) = \sum_a \left(A'_a \cos a + B'_a \sin a\right) e^{-a^2 t} \tag{1.10}$$

The details are problem dependent. If a single series is insufficient to represent all initial and boundary conditions the original problem may be separated into two parts

$$\left\{ \begin{array}{c} q_t^{(1)} = q_{xx}^{(1)} \\ q^{(1)}(x,0) = f(x) \\ q^{(1)}(0,t) = 0, \ q^{(1)}(1,t) = 0 \end{array} \right. \qquad \left\{ \begin{array}{c} q_t^{(2)} = q_{xx}^{(2)} \\ q^{(2)}(x,0) = 0 \\ q^{(2)}(0,t) = g_0(t), \ q^{(2)}(1,t) = g_1(t) \end{array} \right. \tag{1.11}$$

The first problem has simple boundary conditions and the true initial condition and the second has a simple initial condition and the true boundary conditions. The solution to the initial problem is then given by $q = q^{(1)} + q^{(2)}$.

2. Fourier Transform

Taking the $L \to \infty$ limit of (1.3) leads to

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} f(y) \cos \xi(y-x) \ dy \tag{2.1}$$

We can write $\cos \xi(y-x) = \left(\exp\left[i\xi(y-x)\right] + \exp\left[-i\xi(y-x)\right]\right)/2$ and obtain

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} f(y) e^{i\xi(x-y)} \ dy \tag{2.2}$$

whenever the integration operations above are well defined (e.g. when f is absolutely integrable on $\mathbb{R}$). After the y integration a function of ξ, x is obtained which is again integrated to give $f(x)$. This suggests looking at the functions obtained in the intermediate step

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-i\xi y} dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx \ . \tag{2.3}$$

This is called the *Fourier transform* of f. We also have from (2.2)

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi \ . \tag{2.4}$$

We say that f is the *inverse Fourier transform* of $\hat{f}$. The direct and inverse Fourier transforms are integral operators usually denoted by $\mathcal{F}$, $\mathcal{F}^{-1}$ respectively

$$\hat{f} = \mathcal{F}\ f,\ f = \mathcal{F}^{-1}\ \hat{f} \ . \tag{2.5}$$

The transformation is linear

$$f = a\ g + b\ h,\ \hat{f} = a\ \mathcal{F}g + b\ \mathcal{F}h \tag{2.6}$$

with a, b scalars.

An important class of functions for which the Fourier transform is well defined is the L^2 functions f for which

$$\|f\|_2 = \int_{-\infty}^{\infty} |f(x)|^2 dx \tag{2.7}$$

is finite. An important property of the Fourier transform is that it preserves the L^2 norm of a function, i.e.

$$\|f\|_2 = \left\|\hat{f}\right\|_2 \ . \tag{2.8}$$

An illuminating analogy is with vectors v in $\mathbb{R}^n$. Linear transformations of vetors in $\mathbb{R}^n$ are given by matrices A. Most matrices do not preserve the norm of a vector, i.e. in general $\|v\| \neq \|Av\|$. But some special matrices do preserve the norm; examples are rotation and reflection matrices. These basically allow us to look at vectors from various angles and in some orientations the vector becomes especially simple. For example by a succession of rotations we can transform a vector to be parallel to one of the axes in $\mathbb{R}^n$. In an analogous manner, Fourier transforms also allow us to look at functions from various viewpoints from which further insight is possible. We often speak of looking at a function in *real space* or in *wavenumber* or *Fourier space* referring to f or $\hat{f}$, respectively.

An immediate benefit is when we consider PDE's. Note that the Fourier transform is a linear combination of function values amplified by $e^{i\xi x}$. The reason why this particular weighting is useful for PDE's is that it is an eigenfunction of the differentiation operator

$$\partial_x e^{i\xi x} = i\xi e^{i\xi x} \ . \tag{2.9}$$

We can read the above relation as saying that of all the functions we might conceive, for one particular function $e^{i\xi x}$ the differentiation operator reduces to a simple multiplication with a scalar. The benefit of using Fourier transform techniques is that the reduction of differentiation to multiplication carries over to all functions when we look at them in Fourier space. Taking the differential of the Fourier representation of f we obtain

$$f_x = \partial_x f = \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (i\xi)\, \hat{f}(\xi) e^{i\xi x} d\xi \tag{2.10}$$

From this we can deduce that the Fourier transform of f_x is given by

$$\hat{f}_x = i\xi\, \hat{f}. \tag{2.11}$$

3. Fourier solution of common linear PDE's

Using the properties of the Fourier transform it is easy to solve linear PDE's. Let us do this for our basic model problems.

3.1. Advection equation. Taking the Fourier transform with respect to x of

$$q_t + u q_x = 0 \tag{3.1}$$

gives

$$\hat{q}_t + i\xi u \hat{q} = 0 \tag{3.2}$$

which is an ODE in t for $\hat{q}$. The solution is

$$\hat{q}(\xi, t) = \hat{q}(\xi, 0) e^{-i\xi u t} \tag{3.3}$$

which immediately gives $q(x, t0$

$$q(x,t) = \mathcal{F}^{-1}\ \hat{q} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{q}(\xi, t) e^{i\xi x} d\xi \tag{3.4}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{q}(\xi, 0) e^{i\xi(x-ut)} d\xi \tag{3.5}$$

The initial condition $q(x, t = 0) = f(x)$ gives $\hat{q}(\xi, 0) = \mathcal{F}\ f = \hat{f}$. If we write the Fourier representation of f at $x - ut$ we obtain

$$f(x - ut) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi(x-ut)} d\xi \tag{3.6}$$

so the solution to the advection equation is

$$q(x, t) = f(x - ut)\ . \tag{3.7}$$

3.2. Heat equation. We apply the same procedure to

$$q_t = q_{xx} \tag{3.8}$$

to obtain

$$\hat{q}_t = -\xi^2 \hat{q} \tag{3.9}$$

which can be integrated to give

$$\hat{q}(\xi, t) = \hat{q}(\xi, 0) e^{-\xi^2 t}\ . \tag{3.10}$$

Using this Fourier transform we obtain

$$q(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{q}(\xi, 0) e^{-\xi^2 t} e^{i\xi x} d\xi \tag{3.11}$$

It is possible to explicitly evaluate this integral, but we focus instead on interpretation of the behavior of the solution. The formula shows that as time t increases each Fourier mode decays exponentially. The rate of decay is proportional to the square of the wavenumber. Hence, rapid variations in the initial conditions are more quickly attenuated by the heat equation.

4. Von Neumann stability analysis

An important application of Fourier analysis to numerical solutions of PDE's is the ability to quickly determine stability criteria for finite difference schemes. After a finite difference discretization one works with the values of the unknown at grid points at various time levels. Consider the one-dimensional case with a the grid points $x_j = jh$, $h = 1/(M+1)$. Outside of the definition domain $[0,1]$ the grid function is prolonged by periodicity. We shall call the values $\left\{Q_j^n\right\}_{j=0,1,\dots M}$ a *grid function*. The $\left\{Q_j^n\right\}$ can be represented by a Fourier integral

$$Q_j = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} \hat{Q}^n\left(\xi\right) e^{i\xi jh} d\xi \tag{4.1}$$

Note that the wavenumber integral does not go over all possible values, but is restricted to the wavenumbers that can be resolved by the grid. The Fourier transform of the grid function is given by

$$\hat{Q}^n(\xi) = \frac{h}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} Q_j^n e^{-i\xi jh} \ . \tag{4.2}$$

Parseval's relation

$$\left\|\left\{Q_j^n\right\}\right\|_2 = \left\|\hat{Q}^n(\xi)\right\| \tag{4.3}$$

assures us that we can look at the magnitude of the grid function either in Fourier space or in real space. It is typically quite easy to determine a relation between the coefficients $\hat{Q}^n$ in Fourier space for finite difference discretizations of linear PDE's. Once this is done we can compute the amplification ratio

$$G(\xi) = \frac{\hat{Q}^{n+1}(\xi)}{\hat{Q}^n(\xi)} \ . \tag{4.4}$$

This ratio shows the growth of various solution components including the truncation or arithmetic errors. We want these to be kept under control so we shall impose

$$|G(\xi)| \le 1 \tag{4.5}$$