CHAPTER 6

Finite difference methods for hyperbolic equations

1. Scalar equations

1.1. Constant velocity advection in one dimension. The simplest example of a hyperbolic equation is the constant velocity advection equation

$$q_t + u\, q_x = 0 \tag{1.1}$$

with some initial condition $q(x, t = 0) = q_0(x)$. The equation can be solved along the entire real axis in x or some portion thereof. In numerical work we always have a finite subdomain which we shall conveniently choose as $[0, 2\pi]$ with a view to applying Fourier analysis later on. When using a finite subdomain the question of boundary conditions arises which we shall postpone by considering periodic boundary conditions $q(x + 2\pi, t) = q(x, t)$.

1.1.1. *Exact solution by characteristics.* A first attack on finding the solution to (1.1) is to try to reduce it to a simpler problem. One can ask whether there is any subdomain over which the equation can be cast in a simpler form. For instance we can inquire whether there are any particular curves within the (x,t) plane over which the equation simplifies. A general curve Γ of curvilinear parameter is given by

$$\Gamma :\ x = x(s),\ t = t(s) \tag{1.2}$$

and the infinitesimal change in q when going along Γ is

$$\frac{dq}{ds} = \frac{\partial q}{\partial t}\frac{dt}{ds} + \frac{\partial q}{\partial x}\frac{dx}{ds} \tag{1.3}$$

Comparing (1.3) with (1.1) we see that if we impose

$$\frac{dt}{ds} = 1,\ \frac{dx}{ds} = u \tag{1.4}$$

then by (1.1) we must have that

$$\frac{dq}{ds} = 0\ . \tag{1.5}$$

This means that q is constant along the curves Γ defined by (1.4).

1.1.2. *Finite difference methods.* We can construct numerical methods for (1.1) by the same approaches used for the heat equation.

Semi-discretization. Define a computational grid $x_j = jh,\ h = 2\pi/(M+1)$, $t^n = nk$ with step size h, k in space and time. Define $Q_j(t)$ to be the restriction of $q(x,t)$ to $x = x_j$

$$Q_j(t) = q(x_j, t),\ j = 0, 1, \ldots, M + 1\ . \tag{1.6}$$