CHAPTER 6

Finite difference methods for hyperbolic equations

1. Scalar equations

1.1. Constant velocity advection in one dimension. The simplest example of a hyperbolic equation is the constant velocity advection equation

$$q_t + u\ q_x = 0 \tag{1.1}$$

with some initial condition $q(x, t = 0) = q_0(x)$. The equation can be solved along the entire real axis in x or some portion thereof. In numerical work we always have a finite subdomain which we shall conveniently choose as $[0, 2\pi]$ with a view to applying Fourier analysis later on. When using a finite subdomain the question of boundary conditions arises which we shall postpone by considering periodic boundary conditions $q(x + 2\pi, t) = q(x, t)$.

1.1.1. *Exact solution by characteristics.* A first attack on finding the solution to (1.1) is to try to reduce it to a simpler problem. One can ask whether there is any subdomain over which the equation can be cast in a simpler form. For instance we can inquire whether there are any particular curves within the (x,t) plane over which the equation simplifies. A general curve Γ of curvilinear parameter is given by

$$\Gamma:\ x = x(s),\ t = t(s) \tag{1.2}$$

and the infinitesimal change in q when going along Γ is

$$\frac{dq}{ds} = \frac{\partial q}{\partial t}\frac{dt}{ds} + \frac{\partial q}{\partial x}\frac{dx}{ds} \tag{1.3}$$

Comparing (1.3) with (1.1) we see that if we impose

$$\frac{dt}{ds} = 1,\ \frac{dx}{ds} = u \tag{1.4}$$

then by (1.1) we must have that

$$\frac{dq}{ds} = 0\ . \tag{1.5}$$

This means that q is constant along the curves Γ defined by (1.4).

1.1.2. *Finite difference methods.* We can construct numerical methods for (1.1) by the same approaches used for the heat equation.

Semi-discretization. Define a computational grid $x_j = jh,\ h = 2\pi/(M + 1)$, $t^n = nk$ with step size h, k in space and time. Define $Q_j(t)$ to be the restriction of $q(x,t)$ to $x = x_j$

$$Q_j(t) = q(x_j, t),\ j = 0, 1, \ldots, M + 1\ . \tag{1.6}$$

We can choose some approximation of the x derivative. For instance approximating

$$\frac{dq(x_j,t)}{dx} \cong \frac{\delta}{h} Q_j = \frac{Q_{j+1}(t) - Q_{j-1}(t)}{2h} \tag{1.7}$$

leads to the ODE system

$$\frac{d}{dt}\mathbf{Q} = -\frac{u}{2h} B\mathbf{Q} \tag{1.8}$$

with

$$\mathbf{Q} = \begin{bmatrix} Q_1 & Q_2 & \cdots & Q_M \end{bmatrix}^T \tag{1.9}$$

$$B = \begin{matrix} 0 & 1 & & & & -1 \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 1 \\ 1 & & & & -1 & 0 \end{matrix} \tag{1.10}$$

We can now try various ODE schemes to solve (1.8). Using Euler's method would lead to a FTCS scheme

$$\mathbf{Q}^{n+1} = \mathbf{Q}^n - \frac{uk}{2h} B\mathbf{Q}^n = \left(I - \frac{uk}{2h} B \right) \mathbf{Q}^n \ . \tag{1.11}$$

If instead of Euler's scheme we use the midpoint method we obtain the update formula

$$\mathbf{Q}^{n+1} = \mathbf{Q}^{n-1} - \frac{uk}{h} B\mathbf{Q}^n \tag{1.12}$$

known as the *leap-frog* or *Dufort-Frankel* method.

Full discretization. Instead of the semi-discretization or method of lines approach we can also directly discretize both the space and time derivatives appearing in (1.1). A first-order forward in time, second-order centered in space discretization would lead to the FTCS scheme

$$Q_j^{n+1} = Q_j^n - \frac{uk}{2h}\left(Q_{j+1}^n - Q_{j-1}^n \right) \tag{1.13}$$

A modification of (1.13) of historical relevance is the Lax-Friedrichs scheme

$$Q_j^{n+1} = \frac{1}{2}\left(Q_{j+1}^n + Q_{j-1}^n \right) - \frac{uk}{2h}\left(Q_{j+1}^n - Q_{j-1}^n \right) \ . \tag{1.14}$$

This scheme is obtained by replacing Q_j^n with its arithmetic average using values to the left and to the right. Since the formula has not been derived from discretizations of the derivative which we know to be consistent with the original equation it is useful to determine the truncation error. The exact advection operator is

$$D = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \tag{1.15}$$

and we have $Dq = 0$ according to (1.1). Our approximation of this operator is

$$\tilde{D}q(x_j, t^n) = \frac{q_j^{n+1} - \frac{1}{2}\left(q_{j+1}^n + q_{j-1}^n \right)}{k} + \frac{u}{2h}\left(q_{j+1}^n - q_{j-1}^n \right) \tag{1.16}$$

with $q_j^n = q(x_j, t^n)$. The truncation error is therefore

$$\tau_j^n = \left(\tilde{D} - D\right) q(x_j, t^n) = \frac{1}{k} q_j^{n+1} - \frac{1}{2k}\left(q_{j+1}^n + q_{j-1}^n\right) + \frac{u}{2h}\left(q_{j+1}^n - q_{j-1}^n\right) \tag{1.17}$$

We can now carry out a Taylor's series expansion around (x_j, t^n). To simplify notation q and its derivatives will be understood to be evaluated at (x_j, t^n) if not explicitly shown otherwise

(1.18)

$$\tau_j^n = \frac{1}{k}\left(q + kq_t + \frac{k^2}{2} q_{tt} + \dots\right) - \frac{1}{2k}\left(q + hq_x + \frac{h^2}{2} q_{xx} + \dots + q - hq_x + \frac{h^2}{2} q_{xx} + \dots\right)$$

(1.19)

$$+ \frac{u}{2h}\left(q + hq_x + \frac{h^2}{2} q_{xx} + \frac{h^3}{6} q_{xxx} + \dots - q + hq_x - \frac{h^2}{2} q_{xx} + \frac{h^3}{6} q_{xxx} + \dots\right)$$

which gives

$$\tau_j^n = q_t + \frac{k}{2} q_{tt} + \dots - \frac{h^2}{2k} q_{xx} + \dots + uq_x + \frac{uh^2}{12} q_{xxx} + \dots \tag{1.20}$$

Using (1.1) leads to the leading order error

$$\tau_j^n \cong \frac{k}{2} q_{tt} + \frac{uh^3}{12} q_{xxx} = O(k, h^2) \tag{1.21}$$

The analysis shows that the scheme is first order in time and second order in space. The Lax-Friedrichs scheme is consistent, i.e.

$$\lim_{k,h \to 0} \tau_j^n = 0\ . \tag{1.22}$$

Instead of centered finite differences, other approximations may be introduced. A good choice would be to use one-sided finite differences. This would take into account what we know from the exact solution to the advection equation: information travels along the characteristic lines. It would make sense to use finite differences which have a stencil that mimics this behavior. If $u > 0$ we would use left-sided differences and for $u < 0$ we would use right-sided differences. A first order approximation would be

$$Q_j^{n+1} = \begin{cases} Q_j^n - \dfrac{uk}{h}\left(Q_j^n - Q_{j-1}^n\right) & u \geq 0 \\ Q_j^n - \dfrac{uk}{h}\left(Q_{j+1}^n - Q_j^n\right) & u \leq 0 \end{cases} \tag{1.23}$$

This is known, naturally enough, as the *upwind scheme*.

Taylor series approach. A procedure useful in deriving higher order finite difference approximations for (1.1) is the Taylor series approach. In practical work it is economical to only store two time levels at any given stage in the computation. The general prescription for attaining higher order for the time derivative would involve keeping more terms from the operator series

$$\frac{\partial}{\partial t} = \frac{1}{k}\left(\Delta_+ - \frac{\Delta_+^2}{2} + \frac{\Delta_+^3}{3} - \dots\right)\ . \tag{1.24}$$

This would be incovenient since the time stencil of the scheme would become wider and we would need to store more than two time levels. We can however use (1.1) to convert time derivatives into spatial derivatives

$$q_t = -uq_x \tag{1.25}$$

$$q_{tt} = \frac{\partial}{\partial t}(q_t) = -\frac{\partial}{\partial t}(uq_x) = -u\frac{\partial}{\partial x}(q_t) = u\frac{\partial}{\partial x}(uq_x) = u^2 q_{xx} \tag{1.26}$$

The Taylor series approach can now be applied to obtain as high an order of approximation in time as needed

$$q(t+k) = q + kq_t + \frac{k^2}{2}q_{tt} + \frac{k^3}{6}q_{ttt} + \dots \,. \tag{1.27}$$

As an example, let us construct a second order scheme by truncating

$$q(t+k) \cong q + kq_t + \frac{k^2}{2}q_{tt} \tag{1.28}$$

We now replace the time derivatives with spatial derivative

$$q(t+k) \cong q - ukq_x + \frac{u^2k^2}{2}q_{xx} \tag{1.29}$$

and use second order accurate, centered finite differences to approximate the spatial derivatives. The resulting scheme is

$$Q_j^{n+1} = Q_j^n - \frac{uk}{2h}\left(Q_{j+1}^n - Q_{j-1}^n\right) + \frac{u^2k^2}{2h^2}\left(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n\right) \tag{1.30}$$

This is known as the *Lax-Wendroff scheme*.

Instead of centered finite differences we might want to use one-sided formulas to take into account the direction of the characteristics of (1.1). Let us construct a second order accurate one sided approximation using (**??**). One-sided differences can be obtained to arbitrary order of accuracy using the series

$$\frac{\partial}{\partial x} = \frac{1}{h}\left(\Delta_{x+} - \frac{\Delta_{x+}^2}{2} + \frac{\Delta_{x+}^3}{3} - \dots\right) \tag{1.31}$$

$$= \frac{1}{h}\left(\Delta_{x-} + \frac{\Delta_{x-}^2}{2} + \frac{\Delta_{x-}^3}{3} - \dots\right) \tag{1.32}$$

with the finite difference operators defined by

$$\Delta_{x+}q(x,t) = q(x+h,t) - q(x,t) \tag{1.33}$$

$$\Delta_{x-}q(x,t) = q(x,t) - q(x-h,t) \tag{1.34}$$

Let us assume that $u \geq 0$ and therefore that we will be using backward differences so that the computational stencil mimics the true domain of dependence. A second order accurate approximation of q_x is given by

(1.35)

$$q_x(x_j,t^n) = \frac{\partial q}{\partial x}(x_j,t^n) \cong \frac{1}{h}\left(\Delta_{x-} + \frac{\Delta_{x-}^2}{2}\right) q(x_j,t^n)$$

$$= \frac{1}{h}\left[q(x_j,t^n) - q(x_{j-1},t^n) + \frac{1}{2}\left(q(x_j,t^n) - 2q(x_{j-1},t^n) + q(x_{j-2},t^n)\right)\right] \tag{1.36}$$

$$\cong \frac{1}{h}\left[Q_j^n - Q_{j-1}^n + \frac{1}{2}\left(Q_j^n - 2Q_{j-1}^n + Q_{j-2}^n\right)\right] \tag{1.37}$$

$$= \frac{1}{2h}\left(3Q_j^n - 4Q_j^n + Q_j^n\right) \tag{1.38}$$

The second derivative is obtained from

(1.39)
$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial}{\partial x} = \frac{1}{h}\left(\Delta_{x-} + \frac{\Delta_{x-}^2}{2} + \frac{\Delta_{x-}^3}{3} - \dots\right)\frac{1}{h}\left(\Delta_{x-} + \frac{\Delta_{x-}^2}{2} + \frac{\Delta_{x-}^3}{3} - \dots\right)$$

(1.40)
$$= \frac{1}{h^2}\left(\Delta_{x-}^2 + \Delta_{x-}^3 + \frac{11}{12}\Delta_{x-}^3\right)$$

Note that in (**??**) we have neglected terms of $O(k^3)$. The exact solution of the advection equation is $q(x,t) = q_0(x-ut)$. The $x-ut$ argument sugests that similar step sizes should be used for x and t. This will be confirmed by our stability analysis below. So let us assume that $k = O(h)$. Since q_{xx} already has a k^2 factor in (**??**) we only need an $O(h)$ approximation of q_{xx}. The leading order error term from $k^2 q_{xx}$ will then be of $O(k^2 h) = O(k^3) = O(h^3)$. We can therefore truncate the series (1.40) to just the first term and approximate

$$\frac{\partial^2 q}{\partial x^2}(x_j, t^n) \cong \frac{1}{h^2}\Delta_{x-}^2 q(x_j, t^n) = \frac{1}{h^2}\left[q(x_j,t^n) - 2q(x_{j-1},t^n) + q(x_{j-2},t^n)\right] \quad (1.41)$$

$$\cong \frac{1}{h^2}\left(Q_j^n - 2Q_{j-1}^n + Q_{j-2}^n\right) \quad (1.42)$$

Note that this is different from the procedure used in deriving the Lax-Wendroff scheme where a second order accurate expression of q_{xx} was used. The reason is that in the Lax-Wendroff scheme the computational stencil already included Q_{j-1}^n, Q_j^n, Q_{j+1}^n from the approximation of the first derivative q_x. Since the second order accurate approximation of q_{xx} does not widen the stencil there is no penalty in using the more accurate, second order approximation of q_{xx}. In the one-sided scheme we are deriving here however, using a second order accurate approximation of q_{xx} would involve widening the computational stencil to include Q_{j-3}^n. This increases the arithmetic cost of applying the formula without noticeable gain so we choose to use an $O(h)$ approximation of q_{xx}. Combining the above results we obtain

$$Q_j^{n+1} = Q_j^n - \frac{uk}{2h}\left(3Q_j^n - 4Q_{j-1}^n + Q_{j-2}^n\right) + \frac{1}{2}\left(\frac{uk}{h}\right)^2\left(Q_j^n - 2Q_{j-1}^n + Q_{j-2}^n\right) \quad (1.43)$$

for $u > 0$, which is known as the *Beam-Warming scheme*.

1.1.3. *Stability analysis.* We now turn to the analysis of the stability of the various schemes introduced above. The analysis can be done using the techniques for systems of ODE's or using Von Neumann analysis. We shall carry out both procedures.

Semi-discretized system. The matrix B arising in the semi-discretized approach is skew-symmetric and will have purely imaginary eigenvalues. We can check this by explicitly calculating the eigenvalues. As usual, we guess that

$$\mathbf{W}_p = \left[\begin{array}{ccccccc} e^{iph} & e^{ip2h} & \cdots & e^{ipjh} & \cdots & e^{ipMh} \end{array}\right] \quad (1.44)$$

will be an eigenvector since B discretizes a derivation operator. Computing the j^{th} component of $B\mathbf{W}_p$ we get

$$\left(B\mathbf{W}_p\right)_j = e^{ip(j+1)h} - e^{ip(j-1)h} = 2i\sin ph\; e^{ipjh} = 2i\sin ph\;\left(\mathbf{W}_p\right)_j \quad (1.45)$$

so the eigenvalue associated with $\mathbf{W}_p$ is

$$\lambda_p = 2i\sin ph \quad (1.46)$$