that in the Lax-Wendroff scheme the computational stencil already included Q^n_{j-1}, Q^n_j, Q^n_{j+1} from the approximation of the first derivative q_x. Since the second order accurate approximation of q_{xx} does not widen the stencil there is no penalty in using the more accurate, second order approximation of q_{xx}. In the one-sided scheme we are deriving here however, using a second order accurate approximation of q_{xx} would involve widening the computational stencil to include Q^n_{j-3}. This increases the arithmetic cost of applying the formula without noticeable gain so we choose to use an $O(h)$ approximation of q_{xx}. Combining the above results we obtain

$$Q^{n+1}_j = Q^n_j - \frac{uk}{2h}\left(3Q^n_j - 4Q^n_{j-1} + Q^n_{j-2}\right) + \frac{1}{2}\left(\frac{uk}{h}\right)^2\left(Q^n_j - 2Q^n_{j-1} + Q^n_{j-2}\right) \tag{1.43}$$

for $u > 0$, which is known as the Beam-Warming scheme.

1.1.3. Stability analysis. We now turn to the analysis of the stability of the various schemes introduced above. The analysis can be done using the techniques for systems of ODE's or using Von Neumann analysis. We shall carry out both procedures.

Semi-discretized system. The matrix B arising in the semi-discretized approach is skew-symmetric and will have purely imaginary eigenvalues. We can check this by explicitly calculating the eigenvalues. As usual, we guess that

$$\mathbf{W}_p = \left[\, e^{iph} \quad e^{ip2h} \quad \cdots \quad e^{ipjh} \quad \cdots \quad e^{ipMh} \,\right] \tag{1.44}$$

will be an eigenvector since B discretizes a derivation operator. Computing the j^{th} component of $B\mathbf{W}_p$ we get

$$(B\mathbf{W}_p)_j = e^{ip(j+1)h} - e^{ip(j-1)h} = 2i\sin ph\; e^{ipjh} = 2i\sin ph\; (\mathbf{W}_p)_j \tag{1.45}$$

so the eigenvalue associated with $\mathbf{W}_p$ is

$$\lambda_p = 2i\sin ph \tag{1.46}$$

and is indeed purely imaginary.

To establish the stability region for the FTCS method (??) we use the eigenvalues λ of B in the criterion

$$|1 + z| \le 1 \tag{1.47}$$

with $z = k\lambda$. It is immediately apparent that the scheme will be unconditionally unstable because λ is purely imaginary $\lambda = ai$ so

$$|1 + z| = \sqrt{1 + a^2} > 1 \tag{1.48}$$

for all $a > 0$.

The interval of stability for the midpoint scheme is $\mathrm{Re}\, z = 0$, $|\mathrm{Im}(z)| \le 1$. Here we would have

$$z = -\frac{uk}{h} i \sin ph \tag{1.49}$$

and the method is stable for

$$\left|\frac{uk}{h}\right| \le 1\,. \tag{1.50}$$

Von Neumann analysis. Obtaining analytical expression for the matrices arising from the semi-discretized approach becomes increasingly difficult as we use more accurate approximations of the derivatives in the PDE or study PDE's more complex than the advection equation. Von Neumann analysis is typically simpler to apply. We start by determing the stability region for the FTCS scheme (1.13). Substituting a typical wavemode $Q^n = \hat{Q}^n e^{i\xi jh}$ we obtain

$$\hat{Q}^{n+1} e^{i\xi jh} = \hat{Q}^n e^{i\xi jh} - \frac{uk}{2h}\left(\hat{Q}^n e^{i\xi(j+1)h} - \hat{Q}^n e^{i\xi(j-1)h}\right) . \tag{1.51}$$

The amplification ratio is

$$G = \frac{\hat{Q}^{n+1}}{\hat{Q}^n} = 1 - \frac{uk}{h} i \sin \xi h \tag{1.52}$$

which is always greater than 1

$$|G| \ge 1 . \tag{1.53}$$

Thus the scheme is unconditionally unstable as we expected from the semi-discretized stability analysis done above.

For the Lax-Friedrichs scheme we obtain

$$\hat{Q}^{n+1} e^{i\xi jh} = \frac{1}{2}\left(\hat{Q}^n e^{i\xi(j+1)h} + \hat{Q}^n e^{i\xi(j-1)h}\right) - \frac{uk}{2h}\left(\hat{Q}^n e^{i\xi(j+1)h} - \hat{Q}^n e^{i\xi(j-1)h}\right) \tag{1.54}$$

and the amplification factor is

$$G = \cos \xi h - \frac{uk}{h} i \sin \xi h \tag{1.55}$$

Let us introduce the notation

$$\nu = \frac{uk}{h}, \ \theta = \xi h . \tag{1.56}$$

The stability condition is that

$$|G| = \cos^2 \theta + \nu^2 \sin^2 \theta \le 1 = \cos^2 \theta + \sin^2 \theta \tag{1.57}$$

from where we obtain

$$\left(\nu^2 - 1\right) \sin^2 \theta \le 0 . \tag{1.58}$$

The inequality is satisfied for

$$|\nu| \le 1 . \tag{1.59}$$

The quantity ν that appears repeatedly in analysis of numerical schemes for the advection equation is known as the Courant-Friedrichs-Lewy number or more concisely as the CFL number. We say that the Lax-Friedrichs scheme is stable for CFL numbers up to 1, it being implicitly understood that we're considering the absolute value of the velocity $|u|$. From the stability criterion we obtain a bound on the time step that we can use in the Lax-Friedrichs scheme

$$k \le \frac{h}{|u|} . \tag{1.60}$$

For the Lax-Wendroff scheme the amplification ratio is

$$G = 1 - \nu i \sin \theta + \nu^2 (\cos \theta - 1) \tag{1.61}$$

We have

$$|G| = 1 + 2\nu^2(\cos\theta - 1) + \nu^4(\cos\theta - 1)^2 + \nu^2\sin^2\theta \tag{1.62}$$

$$= 1 - 4\nu^2\sin^2\frac{\theta}{2}\left(1 - \cos^2\frac{\theta}{2}\right) + 4\nu^4\sin^4\frac{\theta}{2} \tag{1.63}$$

The stability condition is $|G| \le 1$ leads to

$$\left(\nu^2 - 1\right)\sin^2\frac{\theta}{2} \le 0 \tag{1.64}$$

so again the domain of stability is

$$|\nu| \le 1\,. \tag{1.65}$$

Lax-Wendroff is a more efficient scheme than Lax-Friedrichs since we obtain $O(h^2, k^2)$ precision as opposed to $O(h, k^2)$ under the same time step restriction $k \le h/|u|$.

Turning now to the one-sided schemes, for upwind when $u > 0$ we have

$$G = 1 - \nu\left(1 - e^{-i\theta}\right) \tag{1.66}$$

$$|G| = 1 - 2\nu(1 - \cos\theta) + \nu^2(1 - \cos\theta)^2 + \nu^2\sin^2\theta \tag{1.67}$$

The stability condition $|G| \le 1$ leads to

$$-2\nu(1 - \cos\theta) + \nu^2(1 - \cos\theta)^2 + \nu^2\sin^2\theta \le 1 \tag{1.68}$$

which can be rewritten in terms of the half-angle $\theta/2$ to give

$$-4\nu\sin^2\frac{\theta}{2} + 4\nu^2\sin^4\frac{\theta}{2} + 4\nu^2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} \le 0 \tag{1.69}$$

and finally

$$\nu(\nu - 1) \le 0 \tag{1.70}$$

so the stability region is again $\nu \le 1$.

For the Beam-Warming scheme we have

$$G = 1 - \frac{\nu}{2}\left(3 - 4e^{-i\theta} + e^{-2i\theta}\right) + \frac{1}{2}\nu^2\left(1 - 2e^{-i\theta} + e^{-2i\theta}\right)\,. \tag{1.71}$$

Notice that as we look at more complicated schemes the amplification factors become increasingly difficult to evaluate analytically. We can however use a numerical evaluation of $G(\nu, \theta)$ to generate plots such as Fig. 2. From the plot we deduce that the stability region is $\nu \le 2$.

1.1.4. Lax equivalence theorem. The importance of establishing consistency and stability for a finite difference scheme for the advection equation is that these two properties guarantee convergence by the Lax equivalence theorem.

Theorem 4. A finite difference scheme for a linear PDE is convergent if the scheme is consistent with the PDE and it is stable.

Convergence means that

$$\lim_{k,h \to 0} Q_j^n = q(x_j, t^n) \tag{1.72}$$

where k, h go to zero in accordance with the stability criterion for the scheme. Convergence is obtained when the scheme is consistent, i.e. the truncation error goes to zero

$$\lim_{k,h \to 0} \tau_j^n = 0 \tag{1.73}$$

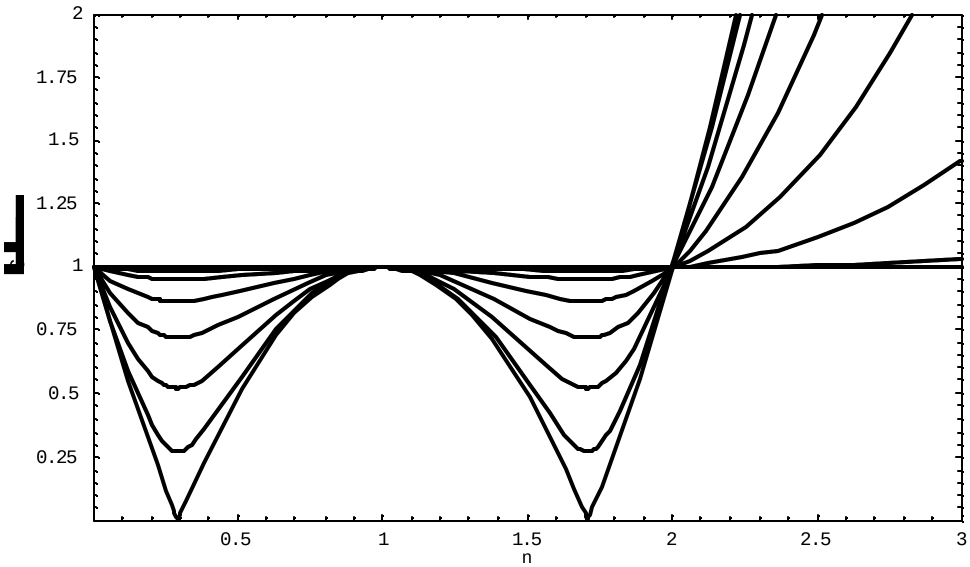

Figure 2. Amplification factor $|G(\nu, \theta)|$ for the Beam-Warming scheme evaluated at $\theta = m\pi/8$, $m = 0, 1, \ldots 16$.

and the step sizes satisfy the stability criterion.

1.1.5. Modified equations. We have established a number of methods for solving the advection equation (1.1). Up to now we have characterized the error of any one scheme by its truncation error. Though indicative of the overall quality of an approximation, the precise nature of the error in the scheme is not apparent. It has proved very fruitful in the development of better methods to more accurately describe how a numerical approximation differs from the exact solution. A question one can ask is whether a given numerical scheme is perhaps a more accurate discretization of another PDE than the one it was originally designed for. Let us exemplify using the upwind scheme for the advection equation with $u > 0$

$$Q_j^{n+1} = Q_j^n - \nu\left(Q_j^n - Q_{j-1}^n\right). \tag{1.74}$$

We know that this scheme is $O(k, h)$ accurate for the equation $q_t + uq_x = 0$. Suppose that the scheme is an exact discretization of some unknown PDE $Ls = 0$ with L an unknown differential operator and $s = s(x, t)$. Then we would have

$$s(x, t + k) = s(x, t) - \nu\left[s(x, t) - s(x - h, t)\right], \tag{1.75}$$

exactly. Let us carry out Taylor series expansion of s around (x, t)

$$s + ks_t + \frac{k^2}{2}s_{tt} + \frac{k^3}{6}s_{ttt} + \ldots = s - \frac{uk}{h}\left(hs_x - \frac{h^2}{2}s_{xx} + \frac{h^3}{6}s_{xxx} - \ldots\right). \tag{1.76}$$

To obtain a more concise notation the function arguments have been dropped. We obtain

$$s_t + us_x = -\frac{k}{2}s_{tt} + \frac{uh}{2}s_{xx} - \frac{k^2}{6}s_{ttt} - \frac{uh^2}{6}s_{xxx} + \ldots \tag{1.77}$$

This is of the form $As = E_{(h,k)}s$ with A the advection operator $A = \partial_t + u\partial_x$ and $E_{(h,k)}$ an operator giving the deviation of the modified equation from the advection