Since $s_{ttx} = u^2 s_{xxx}$ to $O(1)$ we obtain in final

$$s_t + u s_x = -\frac{uh^2}{6}\left(1 - \nu^2\right) s_{xxx}\ . \tag{1.108}$$

The third order derivative now obtained shows that the Lax-Wendroff scheme introduces a dispersive error with different wave numbers traveling at different speeds. As expected, the dispersive error is proportional to h^2 since the Lax-Wendroff scheme is second order. A scheme more accurate than Lax-Wendroff could be obtained by adding a correction term modeling the dispersive error. Since this involves a third-order derivative the stencil of the scheme would become wider by at least one unit thereby entailing more computational work.

1.2. Non-linear scalar equations. We have introduced a number of finite difference methods for the simple constant-velocity advection equation (1.1). Of course, there is hardly much need for a numerical method in order to solve (1.1). Rather we have used (1.1) as a model problem to study the properties of numerical schemes on a simple case. We now proceed to consider more complicated problems and investigate how the methods already derived apply to these problems.

A general first order, hyperbolic scalar equation is given by

$$q_t + u(x,t,q) q_x = \sigma(x,t,q) \tag{1.109}$$

where u may be interpretred as local advection velocity that depends in general upon x, t and q. In a wide range of problems equations of the form

$$q_t + f(q)_x = \sigma(x,t,q) \tag{1.110}$$

arise where f is known as the flux function. If f is differentiable we can write

$$q_t + f_q q_x = \sigma(x,t,q) \tag{1.111}$$

so f_q plays the role of the local advection velocity. Generally u, f depend on q so that the equations become non-linear in q. Equation (1.110) is said to be in conservative form as opposed to (1.109) which is said to be in non-conservative form. Generally we say that a PDE is in conservative form when it can be expressed as the space-time divergence of a vector field. For equation (1.110) the vector field would be $(q, f(q))$ and the space-time divergence is $\nabla_{(t,x)} \cdot = (\partial_t, \partial_x) \cdot$ so another way of writing (1.110) is

$$\nabla_{(t,x)} \cdot (q, f(q)) = 0\ . \tag{1.112}$$

An initial value problem is defined by specifying a solution domain along x and an initial condition $q_0(x)$.

1.2.1. Solution by characteristics. We can solve (1.109) using the moethod of characteristics. We again ask whether there are any special curves Γ within the (x,t) plane on which (1.109) reduces to a simpler form. Along the curves specified by the differential system

$$\frac{dt}{ds} = 1,\ \frac{dx}{ds} = u(x,t,q) \tag{1.113}$$

we do indeed obtain the simpler form

$$\frac{dq}{ds} = \sigma. \tag{1.114}$$

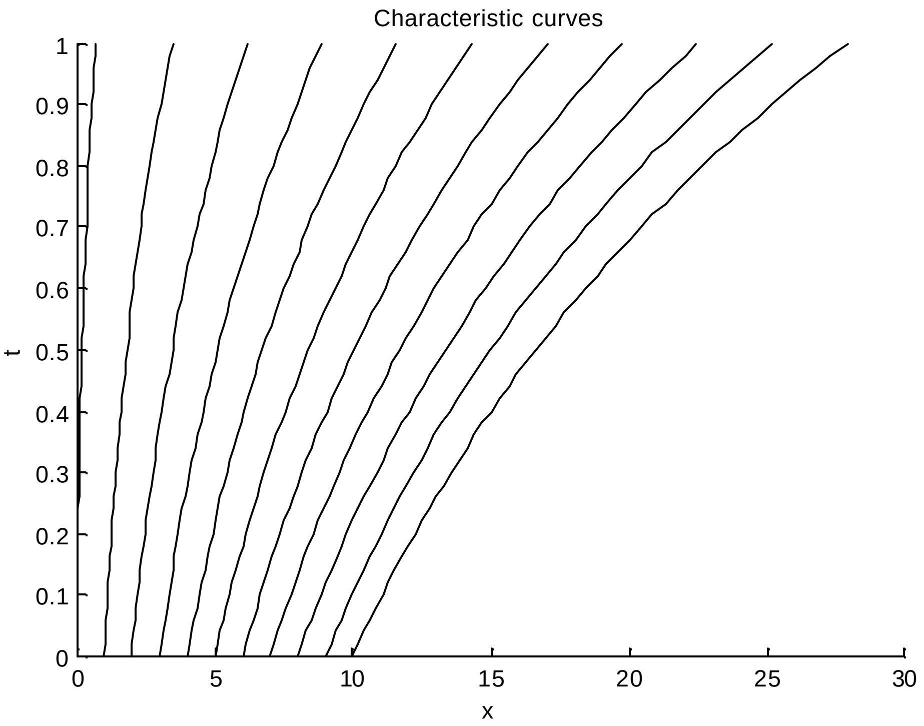


Figure 3. Characteristic curves for $q_t + e^{x+t} q_x = -\beta q$.

The essential difference with respect to the constant-velocity case is that the curves are no longer simple straight lines but depend on x, t and q. Let us consider some examples in order to see the complications involved.

Variable-velocity advection. Consider the equation

$$q_t + u(x,t) q_x = \sigma \tag{1.115}$$

which describes the advection of the unknown field variable q by an imposed velocity field $u(x,t)$. The velocity field is not influenced by q itself; q is said to be a passive tracer. The characteristic curves are given by the ODE

$$\frac{dx}{dt} = u(x,t) \ . \tag{1.116}$$

Note that we are no longer guaranteed that the characteristics exist for all times as they did for the constant-velocity advection equation. This is the case only if u is uniformly Lipschitz.

Example 7. Consider the velocity field

$$u(x,t) = x + t, \tag{1.117}$$

the initial condition

$$q_0(x) = \sin x, \tag{1.118}$$

and the source term

$$\sigma = -\beta q \,. \tag{1.119}$$

The characteristic curves are

$$x(t) = Ce^{t} - t - 1 \tag{1.120}$$

which are shown in Fig. (3). At $t = 0$ the characteristic labeled by C passes through the x coordinate $x_0 = C - 1$. Along each characteristic the variable-velocity advection equation reduces to the ODE

$$\frac{dq}{dt} = -\beta q \tag{1.121}$$

which has the solution $q(x,t) = Ae^{-\beta t}$. We have to determine the constant A from the initial conditions. Through any given point (x,t) there passes the characteristic curve labeled by $C = e^{-t}(x+t+1)$. This particular characteristic curve will intersect the x-axis at $x_0 = C - 1$ and this is the position from which we must take the initial value for q

$$q(x,t) = q_0(e^{-t}(x+t+1) - 1)e^{-\beta t} = \sin(e^{-t}(x+t+1) - 1)e^{-\beta t} \,. \tag{1.122}$$

We have found the solution to the PDE using the simpler expression of the PDE along the characteristics. The solution can be verified by direct substitution in (1.115) and is depicted in Fig. (4). The initial condition is spread out due to the spreading out of the characteristic curves and attenuated due to the source term σ.

Burgers equation. A model equation used extensively in the study of non-linear equations is

$$q_t + qq_x = 0 \tag{1.123}$$

known as the inviscid Burgers equation. It is given in non-conservative form above. In conservative form it becomes

$$q_t + \left(\frac{q^2}{2}\right)_x = 0 \tag{1.124}$$

so the flux function is

$$f(q) = q^2/2 \,. \tag{1.125}$$

The characteristic curves are given by

$$\frac{dx}{dt} = q(x,t) \tag{1.126}$$

and along a characteristic curve $-$ equation (1.123) reduces to

$$\left(\frac{dq}{ds}\right)_{-} = 0 \,, \tag{1.127}$$

i.e. there is no variation in q along the characteristic. This implies that the slope of the each characteristic curve is constant and specified by the initial condition $q(x, t = 0) = q_0(x)$.

The type of difficulties that arise for non-linear equations is immediately apparent from the consideration of simple initial conditions. Consider $q_0(x) = \sin x$. The characteristics are sketched in Fig. 5. The problem is that the characteristic curves cross one another. At such a crossing point it is not apparent what the

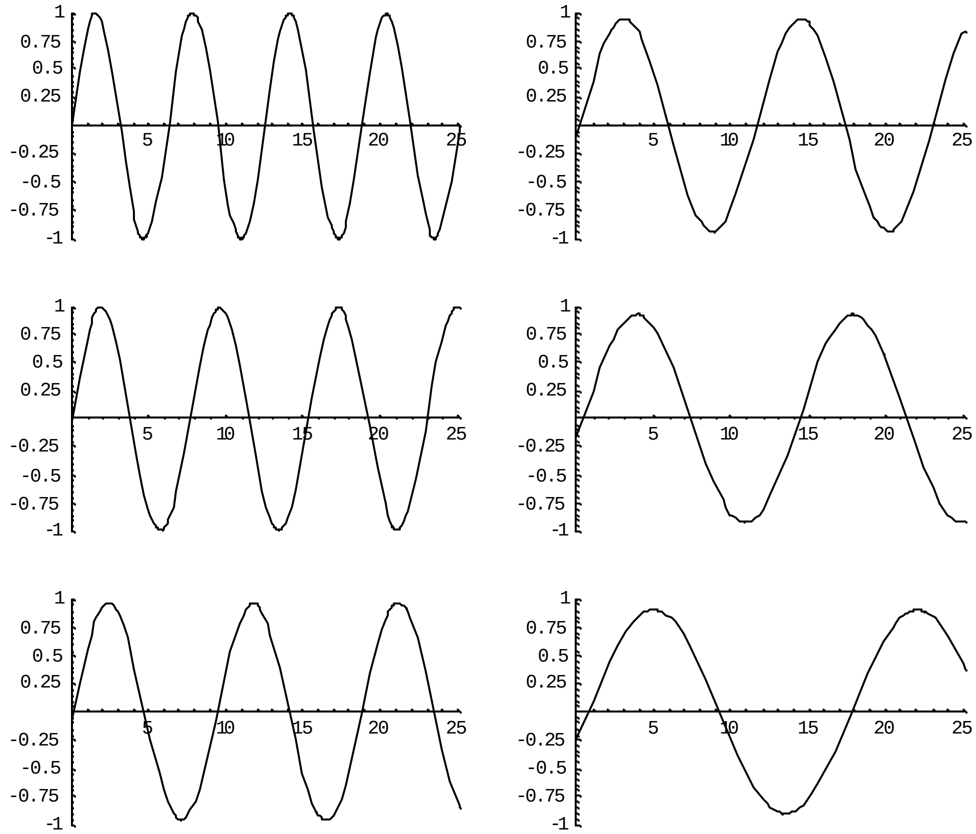


Figure 4. Solution of $q_t + (x + t)q_x = -\beta q$, $q(x, t = 0) = \sin x$ for $\beta = 0.1$ at $t = 0, 0.2, \ldots, 1$.

correct value of q should be since different values are being transported along each of the crossing characteristics.

To get a better idea of what is happening it is useful to simplify the initial condition as much as possible. This leads to the so-called Riemann problem

$$q_0(x) = \begin{cases} q_l & x < 0 \\ q_r & x > 0 \end{cases} \tag{1.128}$$

Let us try to solve Burgers equation for this initial condition.

If $q_l > q_r$ characteristics from $x < 0$ will overtake those from $x > 0$. This will occur on some ray from the origin of equation $x = st$. To the left of this separating ray we will observe the value q_l while to the right we will observe the value q_r. The solution is therefore

$$q(x, t) = \begin{cases} q_l & x < st \\ q_r & x > st \end{cases} \tag{1.129}$$

The initial discontinuity propagates at a velocity s. The discontinuity is called a shock using the language of compressible gas dynamics and s is the shock velocity.

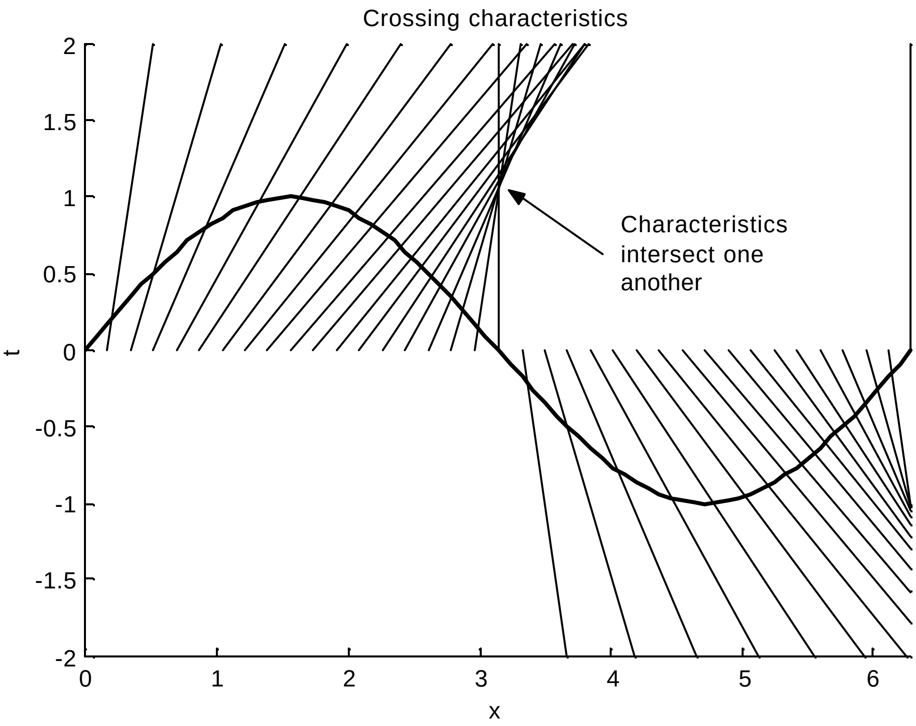


Figure 5. Crossing characteristics for inviscid Burgers equation with initial condition $q_0(x) = \sin x$ (shown in thick line).

The shock velocity can be determined by using the integrating Burgers equation over a domain having the shock as its diagonal $[st_1, st_2] \times [t_1, t_2]$

$$\int_{st_1}^{st_2}\int_{t_1}^{t_2} [q_t + f(q)_x]\,dt\,dx = 0 \tag{1.130}$$

from where

$$s = \frac{f(q_r) - f(q_l)}{q_r - q_l}\,. \tag{1.131}$$

If $q_l < q_r$ two solutions are possible. We can again have the shock solution (1.129) but also the solution

$$q(x,t) = \begin{cases} q_l & x < q_l t \\ x/t & q_l t \le x \le q_r t \\ q_r & x > q_r t \end{cases} \tag{1.132}$$

called a rarefaction solution, again using terms from gas dynamics. This an even worse conundrum, not only can discontinuities arise which invalidate the differentiation operations but multiple solutions seem to be possible. Clearly something is wrong and a way to correct the model that led to equation (??) must be found. From the physical point of view certain effects have been neglected, namely the viscosity of the fluid and we might be led to studying the viscous Burgers equation

$$q_t + qq_x = \nu q_{xx} \tag{1.133}$$

as a remedy to the difficulties encountered. This can be done and leads to smooth solutions with very large gradients in the regions where shocks would have formed for the inviscid Burgers equation. These large gradients are difficult to resolve properly requiring very fine grids, much finer than needed elsewhere in the solution domain. So a way that enables us to still work with the inviscid equation is quite useful.

1.2.2. Weak solutions. The possibility of crossing characteristic curves is indicative with a breakdown of the modeling assumptions that led to a certain hyperbolic PDE. In this situation one must revisit the method by which a certain PDE is derived and consider the validity of all intermediate hypotheses used in the derivation. Burgers equation serves as a useful example. The PDE

$$q_t + f(q)_x = 0 \tag{1.134}$$

with $f = q^2/2$ was proposed as a model for fluid flow in which the quantity q is conserved but being advected by itself. The correct formulation of a conservation principle is through the integral statement

$$\int_{x_1}^{x_2} [q(x,t_2) - q(x,t_1)]\,dx = -\int_{t_1}^{t_2} [f(q(x_2,t)) - f(q(x_1,t))]\,dt \tag{1.135}$$

the one-dimensional expression of (1.9). In this form one can replace

$$q(x,t_2) - q(x,t_1) = \int_{t_1}^{t_2} \frac{\partial q}{\partial t}(x,t)dt \tag{1.136}$$

$$f(q(x_2,t)) - f(q(x_1,t)) = \int_{x_1}^{x_2} \frac{\partial f}{\partial x}dx \tag{1.137}$$

and obtain Burgers equation by going to the limits $t_2 \to t_1$, $x_2 \to x_1$ if the derivatives $\partial q/\partial t$, $\partial f/\partial x$ exist. However one cannot do this if q is discontinuous. In this case only the integral form (1.135) is valid.

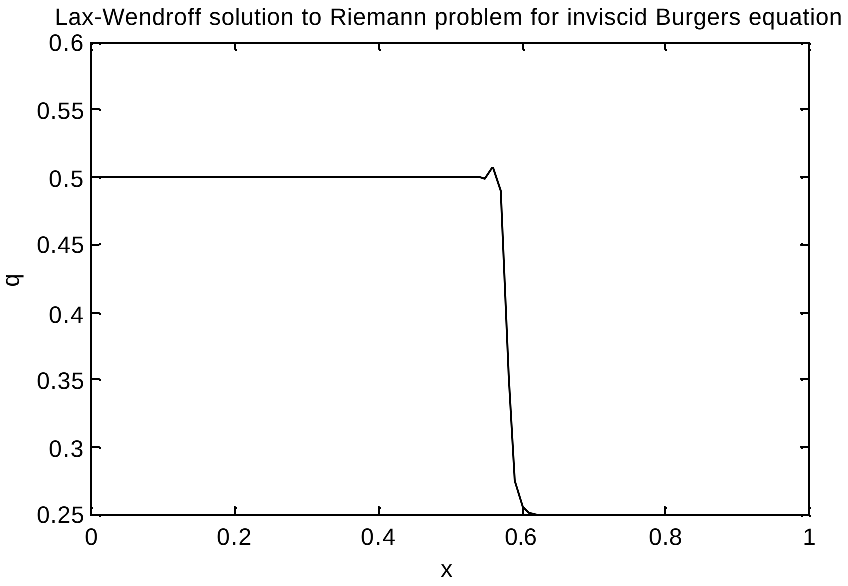
Nonetheless it is typically much more convenient to work with differential equations instead of integral equations. Therefore it is useful to extend the meaning we associate to "q is a solution of a PDE" to cover the case where q might be discontinuous at a few points. This is done through the techniques of the theory of distributions by requiring that q satisfy a certain integral condition. Namely we consider the integral

$$I = \int_0^{\infty}\int_{-\infty}^{+\infty} \phi\,[q_t + f(q)_x]\,dxdt \tag{1.138}$$

with ϕ a smooth function of finite support and impose $I = 0$. Typically we require that ϕ be at least differentiable. We can integrate by parts to obtain

$$\int_0^{\infty}\int_{-\infty}^{+\infty} [\phi_t q + \phi_x f(q)]\,dxdt = -\int_{-\infty}^{+\infty} \phi(x,0)q(x,0)dx\ . \tag{1.139}$$

By this technique all differentiation operations on q have been removed. We say that q is a weak solution of (1.134) if (1.139) is satisfied for all ϕ from some space of test functions such as $\phi \in C^1(\mathbb{R} \times \mathbb{R})$.



1.2.3. Difficulties of finite difference methods for non-linear hyperbolic equations. The possibility of shocks for non-linear hyperbolic equations should alert us to possible difficulties with the finite difference methods we have introduced for the linear advection equation. Since these are based upon Taylor series expansions of $q(x,t)$ and q can be discontinuous, the expansions will break down and not be valid near the discontinuities. Nevertheless, we would expect the methods to be adequate in regions where q is smooth.

Let us see how we would apply the methods to a non-linear equation, taking Burgers equation as an example. One possibility is to interpret q as the local advection velocity u. The upwind method for

$$q_t + qq_x = 0 \tag{1.140}$$

then becomes

$$Q_j^{n+1} = Q_j^n - \begin{cases} Q_j^n \left(Q_j^n - Q_{j-1}^n\right) & \text{if } Q_j^n \geq 0 \\ Q_j^n \left(Q_{j+1}^n - Q_j^n\right) & \text{if } Q_j^n < 0 \end{cases} \tag{1.141}$$

and the Lax-Wendroff methods reads

$$Q_j^{n+1} = Q_j^n - \frac{Q_j^n k}{2h}\left(Q_{j+1}^n - Q_{j-1}^n\right) + \frac{\left(Q_j^n\right)^2 k^2}{2h^2}\left(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n\right). \tag{1.142}$$

Applying this for a Riemann problem leads to a numerical solution similar to the exact shock solution but with oscillations near the shock (Fig. 1.2.3). There is also a smearing of the shock, instead of sharp discontinuity we have a smoothing of q in the vicinity of the shock. Far from the shock the numerical solution is quite good however. This therefore leads to the search for so-called high-resolution algorithms that are able to preserve a high order of accuracy away from discontinuities and also sharply capture discontinuities.

2. Systems of hyperbolic equations

2.1. Linear systems.