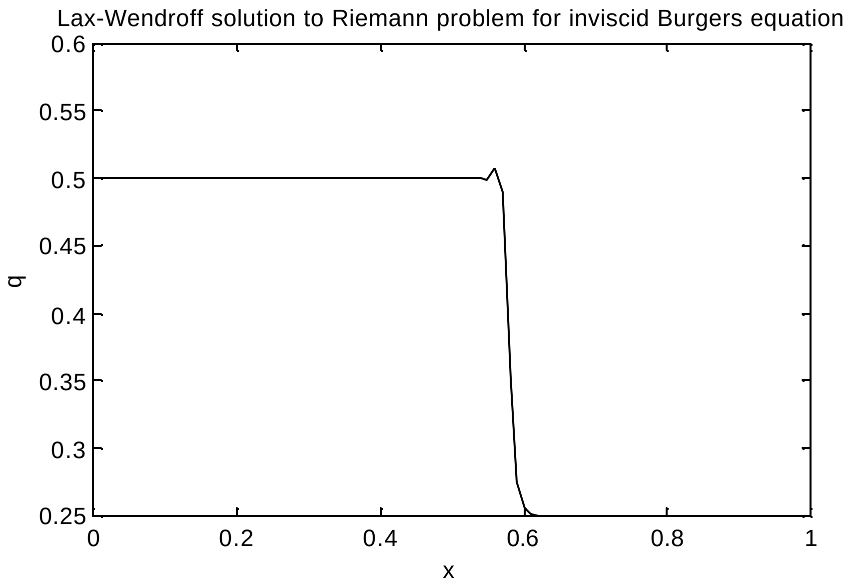

1.2.3. Difficulties of finite difference methods for non-linear hyperbolic equations. The possibility of shocks for non-linear hyperbolic equations should alert us to possible difficulties with the finite difference methods we have introduced for the linear advection equation. Since these are based upon Taylor series expansions of $q(x, t)$ and q can be discontinuous, the expansions will break down and not be valid near the discontinuities. Nevertheless, we would expect the methods to be adequate in regions where q is smooth.

Let us see how we would apply the methods to a non-linear equation, taking Burgers equation as an example. One possibility is to interpret q as the local advection velocity u. The upwind method for

$$q_t + qq_x = 0 \tag{1.140}$$

then becomes

$$Q_j^{n+1} = Q_j^n - \begin{cases} Q_j^n \left(Q_j^n - Q_{j-1}^n\right) & \text{if } Q_j^n \geq 0 \\ Q_j^n \left(Q_{j+1}^n - Q_j^n\right) & \text{if } Q_j^n < 0 \end{cases} \tag{1.141}$$

and the Lax-Wendroff methods reads

$$Q_j^{n+1} = Q_j^n - \frac{Q_j^n k}{2h}\left(Q_{j+1}^n - Q_{j-1}^n\right) + \frac{\left(Q_j^n\right)^2 k^2}{2h^2}\left(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n\right). \tag{1.142}$$

Applying this for a Riemann problem leads to a numerical solution similar to the exact shock solution but with oscillations near the shock (Fig. 1.2.3). There is also a smearing of the shock, instead of sharp discontinuity we have a smoothing of q in the vicinity of the shock. Far from the shock the numerical solution is quite good however. This therefore leads to the search for so-called high-resolution algorithms that are able to preserve a high order of accuracy away from discontinuities and also sharply capture discontinuities.

2. Systems of hyperbolic equations

2.1. Linear systems.

2.1.1. Classification of linear systems. Consider now that we are interested in the simultaneous time evolution of a number of quantities

$$q = \begin{bmatrix} q_1 & q_2 & \dots & q_m \end{bmatrix}^T \tag{2.1}$$

that satisfy

$$q_t + \mathsf{A} q_x = 0 \tag{2.2}$$

with A a constant $m \times m$ matrix of real numbers. Such a system is said to be hyperbolic if the eigenvectors of A form a basis for real m-vectors.

Example 8. The second order wave equation is given in canonical form as

$$\phi_{tt} - c^2 \phi_{xx} = 0 . \tag{2.3}$$

It can be reduced to a system of two first-order equations by introducing

$$u = \phi_t ,\ v = \phi_x \tag{2.4}$$

We have

$$u_t - c^2 v_x = 0 \tag{2.5}$$

and since $\phi_{xt} = \phi_{tx}$

$$v_t - u_x = 0 \tag{2.6}$$

In vector form we obtain

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 & -c^2 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u \\ v \end{bmatrix} = 0 \tag{2.7}$$

or

$$q_t + \mathsf{A} q_x = 0 \tag{2.8}$$

with

$$q = \begin{bmatrix} u \\ v \end{bmatrix} ,\ \mathsf{A} = \begin{bmatrix} 0 & -c^2 \\ -1 & 0 \end{bmatrix} \tag{2.9}$$

The eigenvalues of A are $\lambda_{1,2} = \pm c$ and the eigenvectors are

$$r_1 = \begin{bmatrix} c \\ 1 \end{bmatrix} ,\ r_2 = \begin{bmatrix} -c \\ 1 \end{bmatrix} \tag{2.10}$$

The eigenvectors are independent for $c \neq 0$ and therefore they form a basis for the space of real 2-vectors. The system (2.8) is hyperbolic.

Example 9. Applying the same procedure to the Laplace equation

$$\phi_{tt} + \phi_{xx} = 0 \tag{2.11}$$

leads to the matrix

$$\mathsf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2.12}$$

whose eigenvalues are $\lambda_{1,2} = \pm i$ and eigenvectors are

$$\begin{bmatrix} i \\ 1 \end{bmatrix} ,\ \begin{bmatrix} -i \\ 1 \end{bmatrix} \tag{2.13}$$

These have complex values and are not a basis for real 2-vectors. The system (2.12) is not hyperbolic, it is elliptic.

2.1.2. Solution by method of characteristics and reduction to diagonal form. For hyperbolic systems we can apply a procedure similar to that used for systems of ODE's. We can write

$$A = T\Lambda T^{-1} \tag{2.14}$$

with

$$\Lambda = diag\,\{\lambda_1, \lambda_2, \ldots, \lambda_m\} \tag{2.15}$$

$$T = [r_1\, r_2\, \cdots\, r_m] \tag{2.16}$$

$$A r_j = \lambda_j r_j,\; j = 1, 2, \ldots, m\;. \tag{2.17}$$

and write

$$q_t + T\Lambda T^{-1} q_x = 0\;. \tag{2.18}$$

Introducing the notation

$$w = T^{-1} q \tag{2.19}$$

we obtain

$$w_t + \Lambda w_x = 0\;. \tag{2.20}$$

Since Λ is a diagonal matrix, the equations of the original system have been decoupled and we can write the scalar j^{th} component equation

$$w_t^{(j)} + \lambda_j w_x^{(j)} = 0\;, \tag{2.21}$$

for $j = 1, 2, \ldots, m$. These are now simple constant-velocity advection equations for which we know the solution

$$w^{(j)}(x,t) = w_0^{(j)}(x - \lambda_j t) \tag{2.22}$$

with $w_0^{(j)}$ given by the initial conditions on q

$$w_0 = T^{-1} q_0\;. \tag{2.23}$$

The value of each individual component of $w^{(j)}$ is constant along the family of characteristics $x - \lambda_j t = C_j$. Therefore w are known as the conservative variables. From a knowledge of the conservative variable solution we can recover the solution for the original variables

$$q = Tw\;. \tag{2.24}$$

2.1.3. Finite difference methods. The finite difference methods derived for the constant-velocity advection equation can be applied formally to hyperbolic systems also. For example, the Lax-Wendroff scheme is

$$Q_j^{n+1} = Q_j^n - \frac{k}{2h} A \left(Q_{j+1}^n - Q_{j-1}^n \right) + \frac{k^2}{2h^2} A^2 \left(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n \right) \tag{2.25}$$

There are some new features though due to the fact that there is no longer just a single "advection" or characteristic velocity. Let us try to apply the upwind method to the system (2.8). It is not apparent what the upwind direction should be for q. We can ascertain this for the conservative variables though. We have $\lambda_1 = -c$, $\lambda_2 = c$,

$$r_1 = \begin{pmatrix} c \\ 1 \end{pmatrix},\; r_1 = \begin{pmatrix} -c \\ 1 \end{pmatrix} \tag{2.26}$$

$$T = \begin{bmatrix} c & -c \\ 1 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} \frac{1}{2c} & \frac{1}{2} \\ -\frac{1}{2c} & \frac{1}{2} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -c & 0 \\ 0 & c \end{bmatrix} \tag{2.27}$$

and the conservative variable system is

$$\frac{\partial}{\partial t} \begin{bmatrix} w^{(1)} \\ w^{(2)} \end{bmatrix} + \begin{bmatrix} -c & 0 \\ 0 & c \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} w^{(1)} \\ w^{(2)} \end{bmatrix} = 0 . \tag{2.28}$$

This system can be discretized in an upwind manner and we obtain the scheme

$$\left(W^{(1)}\right)_j^{n+1} = \left(W^{(1)}\right)_j^n + \frac{ck}{h}\left[\left(W^{(1)}\right)_{j+1}^n - \left(W^{(1)}\right)_j^n\right] \tag{2.29}$$

$$\left(W^{(2)}\right)_j^{n+1} = \left(W^{(2)}\right)_j^n - \frac{ck}{h}\left[\left(W^{(2)}\right)_j^n - \left(W^{(2)}\right)_{j-1}^n\right] \tag{2.30}$$

In matrix form this reads

$$W_j^{n+1} = (1-\nu)\, W_j^n + \mathrm{C} W_{j-1}^n + \mathrm{D} W_{j+1}^n \tag{2.31}$$

with

$$\nu = \frac{ck}{h}, \quad \mathrm{C} = \begin{bmatrix} 0 & 0 \\ 0 & \nu \end{bmatrix}, \quad \mathrm{D} = \begin{bmatrix} \nu & 0 \\ 0 & 0 \end{bmatrix} \tag{2.32}$$

Multiplying by T leads to

$$Q_j^{n+1} = (1-\nu)\, Q_j^n + \mathrm{TCT}^{-1} Q_{j-1}^n + \mathrm{TDT}^{-1} Q_{j+1}^n \tag{2.33}$$

$$\mathrm{TCT}^{-1} = \frac{\nu}{2} \begin{bmatrix} 1 & -c \\ -\frac{1}{c} & 1 \end{bmatrix}, \quad \mathrm{TDT}^{-1} = \frac{\nu}{2} \begin{bmatrix} 1 & c \\ \frac{1}{c} & 1 \end{bmatrix} \tag{2.34}$$

This is the upwind scheme for the system (2.8).

2.2. Non-linear systems.

2.2.1. Classification. Non-linear systems are written as

$$q_t + f(q)_x = 0 \tag{2.35}$$

with

$$q = \begin{bmatrix} q_1 & q_2 & \ldots & q_m \end{bmatrix}^T, \quad f = \begin{bmatrix} f_1 & f_2 & \ldots & f_m \end{bmatrix}^T \tag{2.36}$$

The classification of non-linear systems is made in accordance with the properties of the Jacobian of f with respect to q

$$f_q = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \cdots & \frac{\partial f_1}{\partial q_m} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \cdots & \frac{\partial f_2}{\partial q_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial q_1} & \frac{\partial f_m}{\partial q_2} & \cdots & \frac{\partial f_m}{\partial q_m} \end{bmatrix} \tag{2.37}$$

If the eigenvectors of f_q form a basis for q-vectors the system is said to be hyperbolic, otherwise it is elliptic or parabolic. Note that in this case the eigenvectors typically depend on the variables q themselves so that the same system of equations may be hyperbolic in some regions and elliptic in others. The classification of PDE's as hyperbolic, parabolic and elliptic may be more familiar from the classification of second order equations. Let us show the equivalence of the two usages.

The canonical elliptic second order PDE is the Laplace equation

$$\phi_{tt} + \phi_{xx} = 0 \,. \tag{2.38}$$

We reduce it to a system of first-order PDE's by introducing $u = \phi_t,\ v = \phi_x$. The Laplace equation states $u_t + v_x = 0$ and we also have $u_x = v_t$ by the equality of mixed derivatives. These two relations can be written in matrix form as

$$q_t + \mathrm{A} q_x = 0 \tag{2.39}$$

$$q = \begin{bmatrix} u \\ v \end{bmatrix},\ \mathrm{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = 0 \tag{2.40}$$

The matrix A has the eigenvalues $\lambda_{1,2} = \pm i$ and eigenvectors $r_{1,2} = \begin{bmatrix} \pm i & 1 \end{bmatrix}$. The eigenvectors $r_{1,2}$ do not form a basis for two-component real vectors such as q so the system is classified as elliptic in accord with the second-order Laplace equation's classification.

The canonic hyperbolic second order PDE is the wave equation

$$\phi_{tt} - \phi_{xx} = 0 \,. \tag{2.41}$$

Following the same procedure we arrive at the study of the eigensystem of

$$\mathrm{B} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \tag{2.42}$$

which is given by $\lambda_{1,2} = \pm 1$, $r_{1,2} = \begin{bmatrix} \pm 1 & 1 \end{bmatrix}$. The eigenvectors now do form a basis for two-component real vectors and the system is classified as hyperbolic as expected from the wave equation.

Finally, the typical parabolic equation is

$$\phi_x = \phi_{tt} \tag{2.43}$$

for which we denote $u = \phi_t$ to obtain

$$q_t + C q_x = \sigma \tag{2.44}$$

with

$$q = \begin{bmatrix} \phi \\ u \end{bmatrix},\ \mathrm{C} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ \sigma = \begin{bmatrix} u \\ 0 \end{bmatrix}. \tag{2.45}$$

The eigensystem of C is $\lambda_{1,2} = 0$, $r_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $r_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}$ which does not form a basis for two-component real vectors. Note that in this case the rank of C is less than the dimension of the system; this is characteristic of parabolic equations.

2.2.2. Solution by characteristics. Let A be the Jacobian matrix for a non-linear hyperbolic system

$$q_t + \mathrm{A}(q) q_x = 0, \tag{2.46}$$

with q a vector with m components. By the definition of a hyperbolic system we now that A can be represented as

$$\mathrm{A} = \mathrm{T} \Lambda \mathrm{T}^{-1} \,. \tag{2.47}$$

The difference with respect to the linear system case is that the matrices T, Λ are no longer constant but depend on q and hence on (x, t). Nevertheless, we can follow the same procedure of reduction to characteristic form locally for some neighborhood of a point (x_0, t_0) where $q(x, t) = q_0$. We can write

$$q(x, t) = q_0 + \tilde{q}(x, t) \tag{2.48}$$

where $\tilde{q}$ is the perturbation from the value q_0. System (2.46) can now be written

$$\tilde{q}_t + A_0 \tilde{q}_x = 0 \,, \tag{2.49}$$

from where we obtain

$$\tilde{w}_t + \Lambda_0 \tilde{w}_x = 0 \tag{2.50}$$

with the perturbation characteristic variables given by

$$\tilde{w} = T^{-1} \tilde{q}_0 \,. \tag{2.51}$$

The characteristic system (2.50) leads to the ODE's

$$\frac{d\tilde{w}^{(i)}}{ds_i} = 0, i = 1, \ldots, m \,. \tag{2.52}$$

where d/ds_i indicates the derivative along the i^{th} characteristic direction whose slope is given by the λ_{0i} eigenvalue of A_0

$$\frac{d}{ds_i} = \frac{\partial}{\partial t} + \lambda_{0i} \frac{\partial}{\partial x} \,. \tag{2.53}$$

A solution to (2.46) can be found by locally solving the ODE's (2.52). This is the method of characteristics for non-linear hyperbolic systems.