CHAPTER 8

Equations of mixed type

1. Splitting methods

We have determined numerical methods suitable for various types of PDE's such as diffusion modeled by a parabolic equation or advection modeled by a hyperbolic equation. In very many applications multiple effects are present. A typical example would be the advection-diffusion equation

$$q_t + uq_x + vq_y = \alpha\left(q_{xx} + q_{yy}\right) \ . \tag{1.1}$$

The question naturally arises what to do when faced with the solution of such mixed equations. We could develop methods for each new class encountered or combine the methods already developed for simpler equations. The development of new methods for a certain problem class follows the general procedure introduced so far: a discretization is proposed and then stability and accuracy issues are studied.

We shall show however that it is also possible to combine methods developed for simple PDE's in order to obtain schemes for more complicated problems. A general abstract framework is quite useful in this context. Let A, B be any two operators that act upon the unknown function q. We assume q satisfies sufficient regularity hypothesis for the problem at hand. We are interested in whether a solution to the combined problem

$$\partial_t q = (\mathsf{A} + \mathsf{B})q \tag{1.2}$$

can be obtained by schemes suited to the simpler problems

$$\partial_t q = \mathsf{A}q \tag{1.3}$$

$$\partial_t q = \mathsf{B}q \tag{1.4}$$

Formally we can solve each of these equations using operator series. The solution to (1.2) is

$$q(t + k) = e^{(\mathsf{A}+\mathsf{B})k} q(t) \tag{1.5}$$

where the exponential of the operators is defined by its series representation

$$e^{(\mathsf{A}+\mathsf{B})k} = I + k(\mathsf{A} + \mathsf{B}) + \frac{k^2}{2!}(\mathsf{A} + \mathsf{B})^2 + \frac{k^3}{3!}(\mathsf{A} + \mathsf{B})^3 + \ldots \tag{1.6}$$

We assume that the series above converge.

The procedure proposed to solve the original problem by breaking it down into two steps is:

(1) Solve $\partial_t q = \mathsf{A}q$ to obtain an intermediate value $q^*(t + k)$
(2) Use the intermediate value as an initial condition to the second step $\partial_t q = \mathsf{B}q$.

This can be expressed formally as

$$q^S(t+k) = e^{\mathsf{B}k} e^{\mathsf{A}k} q(t) \tag{1.7}$$

where the S superscript denotes this procedure as a splitting approximation. The question is what is the error introduced by the split method. We therefore evaluate

$$E(t+k) = \mathsf{E}q(t) = \left(e^{(\mathsf{A}+\mathsf{B})k} - e^{\mathsf{B}k} e^{\mathsf{A}k} \right) q(t) \ . \tag{1.8}$$

The error operator E can be evaluated by taking the series expansions of all terms involved. In doing this we must be careful in the algebraic manipulations since A, B do not necessarily commute. For instance if $\mathsf{A} = \partial_x$ and $\mathsf{B} = \alpha(x)\partial_x$ we have

$$\mathsf{AB} = a'(x)\partial_x + \alpha(x)\partial_{xx} \tag{1.9}$$

but

$$\mathsf{BA} = \alpha(x)\partial_{xx} \ . \tag{1.10}$$

We have

$$e^{(\mathsf{A}+\mathsf{B})k} = I + k(\mathsf{A}+\mathsf{B}) + \frac{k^2}{2}(\mathsf{A}^2 + \mathsf{AB} + \mathsf{BA} + \mathsf{B}^2) + \dots \tag{1.11}$$

$$e^{\mathsf{B}k} e^{\mathsf{A}k} = \left(I + k\mathsf{B} + \frac{k^2}{2}\mathsf{B}^2 + \dots \right) \left(I + k\mathsf{A} + \frac{k^2}{2}\mathsf{A}^2 + \dots \right) \tag{1.12}$$

$$= I + k(\mathsf{A}+\mathsf{B}) + \frac{k^2}{2}(\mathsf{A}^2 + 2\mathsf{BA} + \mathsf{B}^2) + \dots \tag{1.13}$$

The error operator is therefore

$$\mathsf{E} = (\mathsf{AB} - \mathsf{BA})\frac{k^2}{2} + \dots = [\mathsf{A}, \mathsf{B}]\frac{k^2}{2} + \dots \ . \tag{1.14}$$

The quantity $[\mathsf{A}, \mathsf{B}] = \mathsf{AB} - \mathsf{BA}$ is known as the commutator of A, B. We see that the splitting procedure introduces an $O(k^2)$ error in each time step and we take $O(1/k)$ steps so overall it reduces the numerical accuracy to first order irrespective of the accuracy employed in the individual steps.

A way to avoid the degradation of accuracy is to use a slightly more complicated splitting approach known as Strang splitting

$$q^{SS}(t+k) = e^{\mathsf{A}k/2} e^{\mathsf{B}k} e^{\mathsf{A}k/2} q(t) \ . \tag{1.15}$$

A Taylor series expansion of $e^{\mathsf{A}k/2} e^{\mathsf{B}k} e^{\mathsf{A}k/2}$ shows that the leading order error introduced in each time step is $O(k^3)$ for Strang splitting and $O(k^2)$ overall.