

Finite element methods

1. Preliminaries

For a number of applications the restrictions imposed by finite difference or spectral methods with respect to the computational grid are too severe. This is especially the case in structural engineering where the elasticity equations are solved in domains of complicated geometry such as the interior of an automobile engine. A review of the finite difference and spectral methods would show that the reason relatively simple grids are required is that the differential form of the equation is used. Finite volume methods had no such restriction since they used an integral formulation, and indeed complicated geometries may be treated by finite volume methods. Another class of methods which are based upon an integral formulation are the finite element and closely related boundary element methods. We shall concentrate on finite element methods for now.

The basic idea behind the finite element methods is to employ a piecewise local approximation q of the unknown function q that satisfies some PDE of interest. The piecewise local approximation is defined over some general discretization of the domain of definition of q denoted by Ω . Instead of directly using the piecewise local approximation in the PDE we employ a weighted residual formulation. There arises the significant question of how to best relate the integral formulation to the PDE of interest. Once the discretization, piecewise local approximation and integral formulation are determined a system of equations is obtained whose solution gives the complete approximation to the problem of interest. We shall look at each of these components in detail.

1.1. Spatial discretizations. A domain Ω may be discretized into simple elements in very many ways. Nonetheless only a few are typically used in practice. General affine geometry furnishes some guidance for general discretization techniques. We know for instance that any d_i dimensional domain may be expressed as a reunion of simplices

$$(1.1) \quad \Omega = \bigcup_k S_k .$$

Simplices are the simplest continuum geometric entities one can construct in a space of dimension d . For 1D spaces the simplices are line segments. In 2D they are triangles and in 3D they are tetrahedra. The measure of each of these elements is easily determined by the formulas:

(1) line segment in 1D of nodes $\{x_1, x_2\}$

$$(1.2) \quad l = x_2 - x_1 = \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix}$$

(2) triangle in 2D with nodes $f(x_1, y_1), (x_2, y_2), (x_3, y_3)g$

$$(1.3) \quad A = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

(3) tetrahedron in 3D with nodes $f(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)g$

$$(1.4) \quad V = \frac{1}{6} \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

In the above formulas the element measures are given with sign, the sign corresponding to orientation of the nodes. We understand that the positive value is to be taken whenever a true geometric measure (length, area, volume) is required. Simplicia have many attractive theoretical properties, in particular there exists a definition of what an optimal discretization is for a number of PDE's of interest, especially elliptic PDE's such as the Poisson equation. Fig. 1 shows an example of such a discretization.

Another widely used discretization is into generalized polyhedra having $2d$ sides, i.e. line segments in 1D, quadrilaterals in 2D, hexahedra in 3D. These have the advantage of enabling easier organization of programs since there is a natural ordering of the indices identifying each element. Thus discretizations which use these types of elements give rise to structured computational grids, similar to those encountered in finite difference methods whereas discretizations using simplicia lead to unstructured computational grids.

1.2. Piecewise interpolations. Once a discretization scheme for the geometric domain has been established the next step is to define a local approximation of q over the element E . Typically the approximation is an interpolation based upon values Q_j defined somewhere within the element E , but this is not obligatory and other approximations (spectral elements, Chebyshev elements) may be used. The position where the values Q_j are to be defined must be established. A simple choice is the element nodes but again this is not obligatory and the values may be positioned at other points within E . Finally an interpolation scheme must be established such as polynomial interpolation. Let us give some typical examples:

1.2.1. Linear elements in 1D. The element E has two nodes $f x_1, x_2g, x_2 > x_1$. Values representing $q(x)$ are defined at the nodes $f Q_1, Q_2g$. These define a linear polynomial approximation valid over E

$$(1.5) \quad q(x) = \frac{(x - x_1)Q_2 + (x_2 - x)Q_1}{x_2 - x_1} = N_1(x)Q_1 + N_2(x)Q_2$$

The functions $N_1(x), N_2(x)$ have properties reminiscent of the Dirac delta

$$(1.6) \quad N_1(x_1) = 1, N_1(x_2) = 0$$

$$(1.7) \quad N_2(x_1) = 0, N_2(x_2) = 1$$

and are called form functions. The particular ones used here are called the 1D linear form functions and are depicted in Fig. (2)

Figure 1. Example of the discretization into triangles of the domain between a circle and a NACA-0012 airfoil.

1.2.2. Quadratic elements in 1D. The element E has three nodes $\{x_1, x_2, x_3\}$ and the local approximation is

$$(1.8) \quad \varphi(x) = N_1(x)Q_1 + N_2(x)Q_2 + N_3(x)Q_3$$

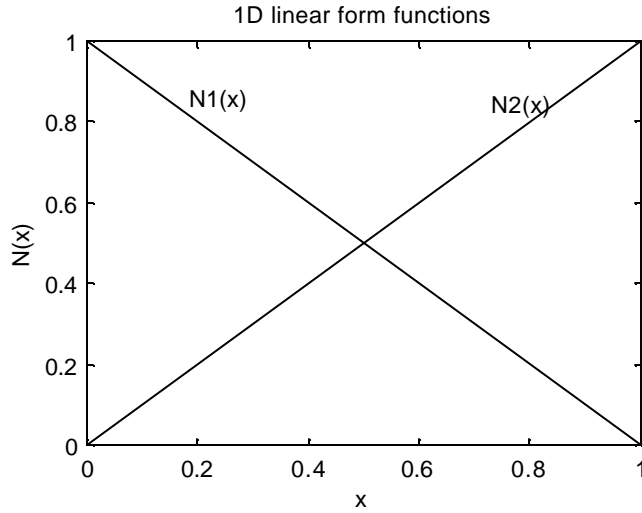


Figure 2. Linear 1D form functions.

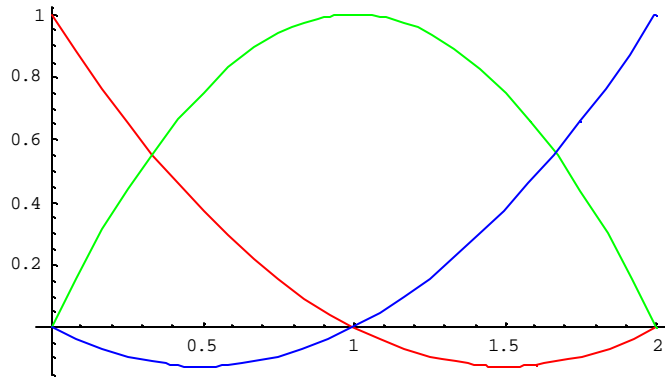


Figure 3. Quadratic element form functions in 1D.

with the form functions

$$(1.9) \quad N_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}$$

$$(1.10) \quad N_2(x) = \frac{(x - x_3)(x - x_1)}{(x_2 - x_3)(x_2 - x_1)}$$

$$(1.11) \quad N_3(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

1.2.3. Linear elements on triangles in 2D. The element E has 3 nodes of coordinates $f(x_1, y_1), (x_2, y_2), (x_3, y_3)$ at which the values Q_1, Q_2, Q_3 are defined. The local approximation of q is given by

$$(1.12) \quad \varphi(x, y) = N_1(x, y)Q_1 + N_2(x, y)Q_2 + N_3(x, y)Q_3$$

with the form functions

$$(1.13) \quad N_1(x, y) = \frac{1}{2A} \begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix} = \frac{1}{2A} (xy_2 - yx_2 - xy_3 + yx_3 + x_2y_3 - x_3y_2)$$

$$(1.14) \quad N_2(x, y) = \frac{1}{2A} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x & x_3 \\ y_1 & y & y_3 \end{vmatrix} = \frac{1}{2A} (yx_1 - xy_1 + xy_3 - yx_3 - x_1y_3 + y_1x_3)$$

$$(1.15) \quad N_3(x, y) = \frac{1}{2A} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x \\ y_1 & y_2 & y \end{vmatrix} = \frac{1}{2A} (xy_1 - yx_1 - xy_2 + yx_2 + x_1y_2 - x_2y_1)$$

Notice how the properties of simplices enable the form functions to be easily determined.

1.2.4. Linear along each direction elements on quadrilaterals in 2D. The element E has 4 nodes $f(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)g$. It is convenient to introduce a local coordinate system (ξ, η) so that the nodes correspond to the local coordinates $(\pm 1, \pm 1)$. The local approximation is then given in the local coordinates by

$$(1.16) \quad \phi(\xi, \eta) =$$

with the form functions

$$(1.17) \quad N_1(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 + \eta)$$

$$(1.18) \quad N_2(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 + \eta)$$

$$(1.19) \quad N_3(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 - \eta)$$

$$(1.20) \quad N_4(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 - \eta)$$

The local transformation of coordinates can also be written in terms of the form functions

$$(1.21) \quad x(\xi, \eta) = \sum_{k=1}^4 N_k(\xi, \eta)x_k, \quad y(\xi, \eta) = \sum_{k=1}^4 N_k(\xi, \eta)y_k$$