with A some differential operator. A function q that directly satisfies (2.27) is called a classical solution. Consider now some space of test functions v and a scalar product defined for the functions q and v. From (2.27) we can derive

$$(\mathrm{A}q, v) = (g, v) \tag{2.28}$$

where $(\cdot,\cdot)$ denotes the scalar product, e.g.

$$(u, v) = \int_a^b u(x)v(x)dx \ . \tag{2.29}$$

In (2.28) we can apply integration by parts to obtain

$$(q, \mathrm{A}^*v) = (g, v) \tag{2.30}$$

where A^* is the adjoint operator of A. This typically enables us to avoid differentiating functions q that might be discontinuous. We can now use (2.30) to determine the unknown coefficients of a finite element approximation

$$\tilde{q}(x) = \sum_e \sum_k Q_k^e N_k^e(x) \tag{2.31}$$

by requiring

$$\sum_e \sum_k Q_k^e (N_k^e(x), \mathrm{A}^*v) = (g, v) \ . \tag{2.32}$$

The only piece missing is how we choose the test functions v. In a Galerkin formulation these are chosen to be the form functions themselves leading to

$$\sum_e \sum_k Q_k^e \left(N_k^e(x), \mathrm{A}^*N_j^e(x) \right) = (g, N_j^e(x)) \ , \tag{2.33}$$

thus defining a linear system

$$\mathrm{A}Q = b \tag{2.34}$$

$$A_{jk} = \left(N_k^e(x), \mathrm{A}^*N_j^e(x) \right) \ . \tag{2.35}$$

2.4. A detailed example. Let us now carry out the steps involved in solving a Poisson equation in 2D using a Ritz formulation and quadrilateral elements. The mathematical statement of the problem is

$$\left\{ \begin{array}{ll} q_{xx} + q_{yy} = g & (x, y) \in \Omega \\ q = b & (x, y) \in \partial\Omega \end{array} \right. \tag{2.36}$$

with the domain $\Omega = [a, b] \times [c, d]$ and $\partial\Omega$ denoting the boundary of Ω on which Dirichlet conditions are given. The element form functions are given by (1.17)-(1.20) and the function f is given by (2.22). The function $I(\tilde{q})$ is

$$I(\tilde{q}) = \int_c^d \int_a^b f(x, y, \tilde{q}, \tilde{q}_x, \tilde{q}_y)dxdy = \int_c^d \int_a^b \left[\frac{1}{2}\left(\tilde{q}_x^2 + \tilde{q}_y^2 \right) - g\tilde{q} \right] dxdy \ . \tag{2.37}$$

The finite element approximation is determined by the chosen form functions and the nodal values $\{Q_k^e\}$. The extremum of $I(\tilde{q})$ is attained when

$$\frac{\partial}{\partial Q_k^e} I(\tilde{q}) = 0 \tag{2.38}$$

which leads to

$$\int_c^d \int_a^b \left[\left(\tilde{q}_x \frac{\partial \tilde{q}_x}{\partial Q_k^e} + \tilde{q}_y \frac{\partial \tilde{q}_y}{\partial Q_k^e} \right) - g \frac{\partial \tilde{q}}{\partial Q_k^e} \right] dxdy = 0 \tag{2.39}$$

Note that

$$\frac{\partial \tilde{q}}{\partial Q_k^e} = N_k^e,\ \frac{\partial \tilde{q}_x}{\partial Q_k^e} = \frac{\partial N_k^e}{\partial x},\ \frac{\partial \tilde{q}_y}{\partial Q_k^e} = \frac{\partial N_k^e}{\partial y} \tag{2.40}$$

so these derivatives no longer contain the unknowns $\{Q_k^e\}$. We thus obtain

$$\sum_e \sum_j \left[\iint \left(\frac{\partial N_j^e}{\partial x}\frac{\partial N_k^e}{\partial x} + \frac{\partial N_j^e}{\partial y}\frac{\partial N_k^e}{\partial y} \right) dxdy \right] Q_k^e = \sum_e \iint g N_k^e \, dxdy \tag{2.41}$$

with k going over all the element nodes. The sum over the elements is typically known as an assembly operation, leading to the computation of the matrix elements

$$A_{jk} = \sum_e \iint \left(\frac{\partial N_j^e}{\partial x}\frac{\partial N_k^e}{\partial x} + \frac{\partial N_j^e}{\partial y}\frac{\partial N_k^e}{\partial y} \right) dxdy \tag{2.42}$$

known as the system stiffness matrix. We can easily compute the elements of this matrix. Analytical computation is possible as in

$$\frac{\partial N_k^e}{\partial x} = \frac{\partial N_k^e}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_k^e}{\partial \eta}\frac{\partial \eta}{\partial x} \tag{2.43}$$

$$\frac{\partial \xi}{\partial x} = \frac{\frac{D(\xi,y)}{D(\xi,\eta)}}{\frac{D(x,y)}{D(\xi,\eta)}} = \frac{1}{J}\begin{vmatrix} 1 & 0 \\ y_\xi & y_\eta \end{vmatrix} = \frac{y_\eta}{J} \tag{2.44}$$

$$\frac{\partial \eta}{\partial x} = \frac{\frac{D(\eta,y)}{D(\xi,\eta)}}{\frac{D(x,y)}{D(\xi,\eta)}} = \frac{1}{J}\begin{vmatrix} 0 & 1 \\ y_\xi & y_\eta \end{vmatrix} = -\frac{y_\xi}{J} \tag{2.45}$$

$$J = \begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix} = x_\xi y_\eta - x_\eta y_\xi \tag{2.46}$$

The $x(\xi,\eta)$ and $y(\xi,\eta)$ dependencies are given by (1.21) so we obtain

$$\frac{\partial x}{\partial \xi} = \sum_{k=1}^{4} \frac{\partial N_k}{\partial \xi} x_k,\ \frac{\partial y}{\partial \xi} = \sum_{k=1}^{4} \frac{\partial N_k}{\partial \xi} y_k \tag{2.47}$$

$$\frac{\partial x}{\partial \eta} = \sum_{k=1}^{4} \frac{\partial N_k}{\partial \eta} x_k,\ \frac{\partial y}{\partial \eta} = \sum_{k=1}^{4} \frac{\partial N_k}{\partial \eta} y_k \tag{2.48}$$

But analytical evaluations are not really required in this case. We can recognize that the integrand in (2.42) is quadratic in (x,y) and that a 4-point Gauss-Legendre quadrature leads to an exact evaluation.