Final Examination

Numerical solution of partial differential equations, I (Course 221)

December 10, 2002

1. The algorithm

$$Q^{n+3} - \frac{11}{6}Q^{n+2} + Q^{n+1} - \frac{1}{6}Q^{n} = \frac{k}{3}f(Q^{n+3}) \qquad (1)$$

with k the time step size and $Q^n \cong q(t^n)$, is being proposed to solve the IVP

$$\frac{dq}{dt} = f(q,t) \qquad (2)$$
$$q(t = 0) = q_0 \qquad (3)$$

(a) Is the algorithm zero-stable? (2 points)

Solution The lhs characteristic polynomial is

$$\rho(\zeta) = \zeta^3 - \frac{11}{6}\xi^2 + \zeta - \frac{1}{6}\ . \qquad (4)$$

We try $\zeta = 1$ (since otherwise the algorithm wouldn't be consistent) and find $\rho(1) = 0$ and arrive at the factorization

$$\rho(\zeta) = (\zeta - 1)(\zeta - \frac{1}{2})(\zeta - \frac{1}{3}) \qquad (5)$$

The roots of $\rho(\zeta)$ are $\zeta_1 = 1$, $\zeta_2 = 1/2$, $\zeta_3 = 1/3$. There is one root of absolute value 1 and two others of absolute value less than 1. The lhs polynomial satisfies the stability condition and the algorithm is zero-stable.

(b) Is the algorithm consistent? (2 points)

Solution The rhs characteristic polynomial is

$$\sigma(\zeta) = \frac{\zeta^3}{3} \qquad (6)$$

The two conditions $\rho'(1) = \sigma(1)$ and $\rho(1) = 0$ are verified and the algorithm is consistent.

(c) Does the algorithm furnish a convergent series of approximations to the exact solution of the IVP? (2 points)

Solution The algorithm is consistent and zero-stable, therefore it is convergent.

(d) Determine the range of step sizes for which the algorithm is absolutely stable. The boundary locus

$$z(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} \qquad (7)$$

is rendered below. (2 points)

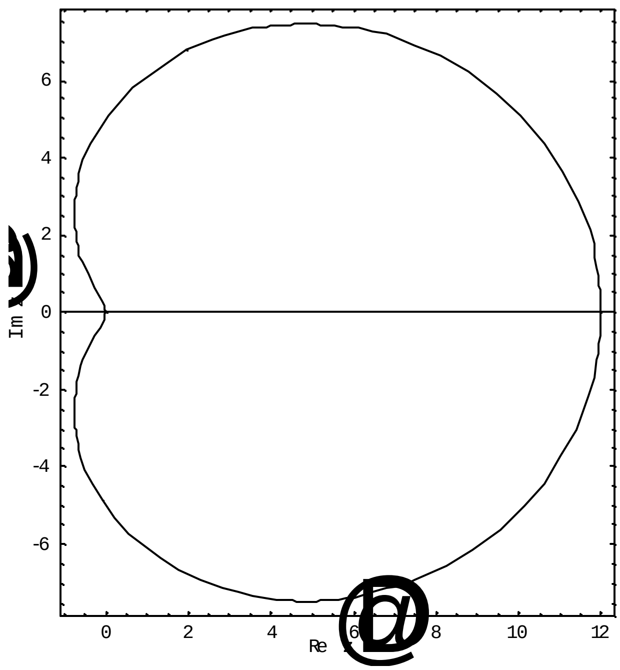

Figure 1:

Solution The characteristic polynomial of the full scheme is

$$\pi(\zeta; z) = \rho(\zeta) - z\sigma(\zeta) = \zeta^3 - \frac{11}{6}\xi^2 + \zeta - \frac{1}{6} - z\frac{\zeta^3}{3} \qquad (8)$$

with $z = k\lambda$ and λ the scalar from the model problem $f = \lambda q$. We use the boundary locus graph to delimit regions in the z-plane. There are two regions. We must find the roots of π in a point in each of these regions. Choosing $z = 30$ leads to the polynomial

$$\pi(\zeta; 30) = 11\zeta^3 - \frac{11}{6}\xi^2 + \zeta - \frac{1}{6} = 11\zeta^2(\zeta - \frac{1}{6}) + \zeta - \frac{1}{6} = (11\zeta^2 + 1)(\zeta - \frac{1}{6}) \qquad (9)$$

with roots that are all less than 1 in absolute value so the region containing $z = 30$ is a region of stability. For the second region one may be tempted to use $z = 3$ and obtain

$$\pi(\zeta; z = 3) = -\frac{11}{6}\xi^2 + \zeta - \frac{1}{6} \qquad (10)$$

but the tempation should be resisted! We find out nothing about the *third* root of the characteristic polynomial by this procedure and the third root exists at all points other than $z = 3$. Rather we should make some estimate on the value of the roots since the analytical Tartaglia formulas are rather cumbersome. Here an obvious observation is that $z = 0$ is on the boundary locus since $\rho(\zeta = 1) = 0$ so the question is what happens to the $\zeta = 1$ root under perturbation by the $\frac{z}{3}\zeta^3$ term. For a very small perturbation the root will still be close to 1 and the only question is whether it is greater or less than one. We can for instance apply Newton's method to

$$f(\zeta) = \zeta^3 - \frac{11}{6}\xi^2 + \zeta - \frac{1}{6} - z\frac{\zeta^3}{3} = 0 \tag{11}$$

to obtain the update formula

$$\zeta^{(1)} = \zeta^{(0)} - \frac{f(\zeta^{(0)})}{f'(\zeta^{(0)})} \tag{12}$$

Taking $\zeta^{(0)} = 1$ as the initial approximation we find

$$\zeta^{(1)} = 1 + \frac{z}{1 - 3z} = 1 + z(1 + 3z + \ldots) \tag{13}$$

since z is small. It is apparent that for $z > 0$ the root is greater than 1 and we ascertain that the second region is a region of instability.

(e) Is the algorithm A-stable, i.e. does the region of stability include the left half plane $\operatorname{Re} z < 0$? (2 points)

Solution No the method is not A-stable since there are points of $\operatorname{Re} z < 0$ in the region of instability, e.g. $\varepsilon + 3i$, small ε.

(f) Is the algorithm L-stable, i.e. does the region of stability include the point at infinity? (2 points)

Solution Yes, the method is L-stable, the point at infinity is within the region of stability.

(g) Describe an appropriate starting procedure for this algorithm, i.e. how one would determine Q^1, Q^2. (2 points)

Solution We must start the algorithm with a method giving Q^1, Q^2 to the same order of precision as the algorithm itself. This is a question in disguise on the truncation error of the algorithm. We compute

$$\tau^{n+3} = \frac{3\left[q(t^{n+3}) - \frac{11}{6}q(t^{n+2}) + q(t^{n+1}) - \frac{1}{6}q(t^n)\right]}{k} - q'(t^{n+3}) \tag{14}$$

by Taylor series expansions around t^{n+3} and find

$$\tau^{n+3} = O(k), \tag{15}$$

i.e. the algorithm is first order accurate. We can start the algorithm with an explicit Euler scheme taking care that the step size used falls within both the absolute stability region for Euler's method and the algorithm here.

(h) The algorithm is implicit. Propose an appropriate explicit formula to obtain a predictor-corrector method. (2 points)

Solution The method is only first order accurate. Any explicit formula will work, including explicit Euler.

2. Suppose the above algorithm is being applied to the ODE system obtained by semidiscretizing the PDE

$$q_t + uq_x = 0 \tag{16}$$

with respect to x using a centered, second order accurate approximation of q_x

$$q_x(x_j,t) \cong \frac{Q_{j+1}(t) - Q_{j-1}(t)}{2h} \ . \tag{17}$$

The PDE is valid on $(x,t) \in [0,2\pi] \times [0,T]$, with $u > 0$ a constant, $q(x+2\pi,t) = q(x,t)$ and $q(x,t=0) = \sin x$.

(a) Write out the update formula you would use to advance your numerical approximation forward in time. (2 points)

Solution We obtain

$$Q_j^{n+3} - \frac{11}{6}Q_j^{n+2} + Q_j^{n+1} - \frac{1}{6}Q_j^n = -\frac{uk}{6h}(Q_{j+1}^{n+3} - Q_{j-1}^{n+3}) \tag{18}$$

or

$$\frac{\nu}{6}Q_{j+1}^{n+3} + Q_j^{n+3} - \frac{\nu}{6}Q_{j-1}^{n+3} = \frac{11}{6}Q_j^{n+2} - Q_j^{n+1} + \frac{1}{6}Q_j^n \tag{19}$$

with $\nu = uk/h$ and j going over all interior points. To advance forward in time we must solve a tridiagonal system at each time step.

(b) Comment on the practicality of the above algorithm. (2 points)

Solution The algorithm will be $O(k,h^2)$. This is better than the $O(k,h)$ accuracy of upwind but worse than the $O(k^2,h^2)$ accuracy of Lax-Wendroff. It involves the solution of a tridiagonal system thereby entailing more work than either of the above methods. It would only be a practical algorithm is the stability region is larger than that for upwind or Lax-Wendroff.

(c) Is the overall algorithm a consistent approximation of the PDE? (2 points)

Solution Yes, the ODE solver part was shown to be consistent in the previous problem and we used a consistent approximation of the x-derivative.

(d) Determine the amplification factor of an arbitrary Fourier mode (von Neumann stability analysis). (2 points)

Solution Plugging in an arbitrary Fourier mode $Q_j^n = e^{ijh\xi}$, $Q_j^{n+m} = G^m e^{ijh\xi}$ leads to

$$(\frac{\nu i}{3}\sin\xi h + 1)G^3 = \frac{11}{6}G^2 - G + \frac{1}{6} \tag{20}$$

The largest amplification factor is the solution of the above cubic equation of greatest absolute value.

(e) Is the overall algorithm stable and if so for what range of step sizes k, h? (2 points)

Solution We've just seen that von Neumann analysis leads to an unwieldy cubic. We can however use what we know of the stability properties of the base ODE scheme. The only problem is to determine the eigenvalues of the rhs operator matrix

$$A = -\frac{uk}{6h}\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & & \ddots & \end{bmatrix} \tag{21}$$

Lest we had forgotten them, it is easy to compute the eigenvalues by guessing that the eigenvector is a discretization of the eigenfunction of ∂_x which we know to be $e^{i\xi x}$

$$\partial_x e^{i\xi x} = i\xi e^{i\xi x} \ . \tag{22}$$

so the eigenvector of A is guessed to have components $e^{ijh\xi}$. This leads to

$$\lambda = -\frac{i}{3}\nu\sin\xi h \tag{23}$$

so the extremal values are $\lambda_1 = -\frac{i}{3}\nu$, $\lambda_1 = +\frac{i}{3}\nu$. A look at the boundary locus graph suggests that we should have

$$\frac{\nu}{3} \lesssim 4 \tag{24}$$

so the stability restriction is

$$\frac{uk}{h} \lesssim 12 \ . \tag{25}$$

This is pretty good; we can use a CFL number of 12 to advance the solution, so the algorithm turns out to be attractive especially for stiff problems.

3. Consider the PDE

$$q_t + (x+t)q_x = 0, \tag{26}$$

and the initial condition

$$q(x, t=0) = \sin x \tag{27}$$

(a) Determine the characteristic curves for the PDE(2 points).

Solution We must solve the ODE system

$$\frac{dt}{ds} = 1 \tag{28}$$

$$\frac{dx}{ds} = x + t \tag{29}$$

or, equivalently

$$\frac{dx}{dt} - x = t \tag{30}$$

The solutino of the homogeneous equation is

$$x(t) = Ce^t. \tag{31}$$

By variation of constants we obtain

$$C' = te^{-t} \tag{32}$$

which can be integrated to give

$$C(t) = -e^{-t}(1+t) + A \tag{33}$$

so the solution is

$$x(t) = -1 - t + Ae^t \tag{34}$$

At $t = 0$ the characterustic curve labeled by A passes through

$$x = -1 + A \tag{35}$$

It's easy to see that the curve $A = 0$ is a straight line separating curves which go the right and those that go the left. Here's a rendering of some of the characteristic curves:

(b) Determine the exact solution at (x, t). (2 points)

Solution Along a characteristic curve we have

$$\frac{dq}{ds} = 0 \tag{36}$$

so to determine the exact solution at (x, t) we must determine the particular characteristic curve which passes through that point

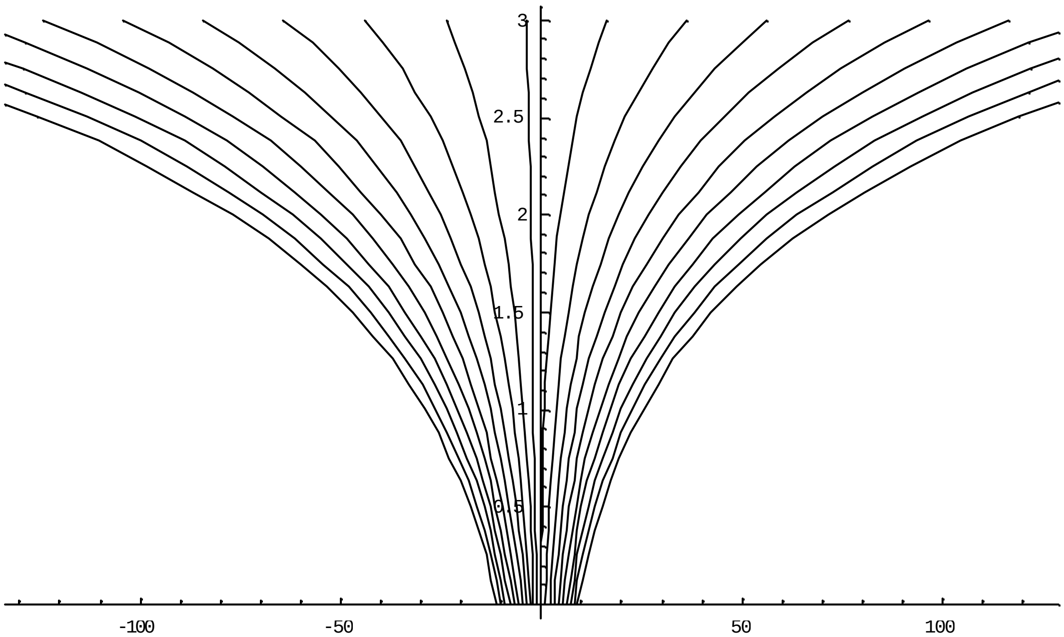

Figure 2:

and its intersect with the x-axis. The characteristic curve passing through (x, t) is labeled by

$$A = (x + t + 1)e^{-t} \tag{37}$$

It intersects the x-axis at $-1 + A$ where the initial condition is $\sin(A - 1)$. The exact solution is therefore

$$q(x, t) = \sin\left[(x + t + 1)e^{-t} - 1\right] \tag{38}$$

and can be verified by substitution in the PDE.

(c) Sketch what you think the solution will look like at a later time (3 bonus points). Motivate your sketch.

Solution The sine wave initial condition gets stretched out by the characteristic curves. Here's a plot superimposing the characteristic curves, $q(x, 0)$ and $q(x, 3)$.

(d) Sketch what you think the numerical solution using the upwind method will look like at a later time (3 bonus points). Motivate your sketch.

Solution Using upwind we would expect the dominant error term to be diffusive in nature so we would expect to see a slightly damped sine wave at later times.

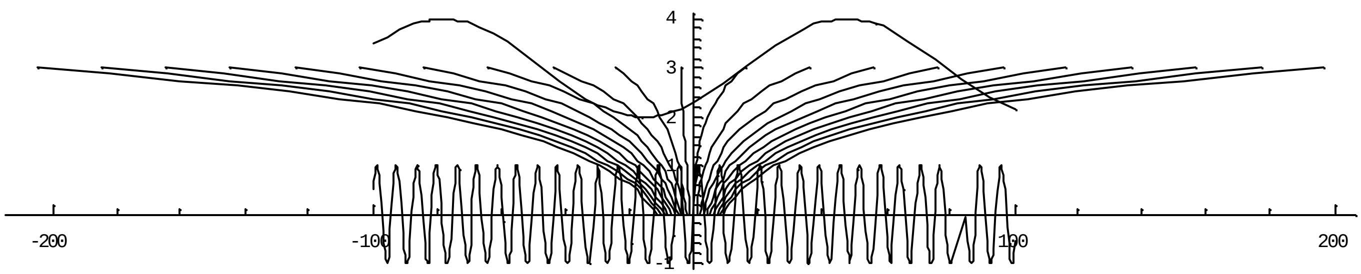

Figure 3:

(e) Sketch what you think the numerical solution using the Lax-Wendroff method will look like at a later time (3 bonus points). Motivate your sketch.

Solution The dominant error term is dispersive, but the initial condition only contains one Fourier mode so there is no wave packet to disperse. The next term in the Lax-Wendroff modified equation is hyper-diffusive so there will be a small attenuation of the sine wave, much less than that in the upwind method.

(f) Explain how you would apply the upwind method to this problem (3 bonus points).

Solution We would have to determine the sign of the local advection velocity

$$u = x + t \tag{39}$$

To the right of the $x = -t$ line we would use

$$Q_j^{n+1} = Q_j^n - \frac{uk}{h}(Q_j^n - Q_{j-1}^n) \tag{40}$$

and to the left we would use

$$Q_j^{n+1} = Q_j^n - \frac{uk}{h}(Q_{j+1}^n - Q_j^n) \tag{41}$$

(g) Give the solution to a Riemann problem for the PDE above. How would you use this solution in a finite volume method? (3 bonus points).

Solution The Riemann problem has the initial condition

$$q(x,0) = \begin{matrix} q_l & x < 0 \\ q_r & x > 0 \end{matrix} \tag{42}$$

so the entire question reduces to looking at the characteristic which passes through $x = 0$. This is the characteristic labeled by $A = 1$

$$x(t) = -1 - t + e^t \tag{43}$$

To the left of this curve we have $q(x,t) = q_l$ at all later times and to the right we have $q(x,t) = q_r$

(h) Write down a spectral method solution of this problem (5 bonus points).

Solution We need to recognize that the advection velocity is space-time dependent. This means that we should evaluate q_x in spectral space, transform back to real space to compute $(x+t)q_x$ and then transform the result to spectral space for time advancement. The scheme would work as follows:

$$\{Q^n\} \to \left\{\hat{Q}^n\right\} \to \left\{ik\hat{Q}^n\right\} \to q_x \cong F^{-1}\left\{ik\hat{Q}^n\right\} \to c = (x+t)q_x \to \hat{c} \to \hat{q}_t + \hat{c} = 0 \tag{44}$$

(i) Write down a finite element method solution of this problem using quadratic elements. (10 bonus points)

Solution An element would be the segment $E_j = [x_j, x_{j+1}]$. We need three nodes per element since we wish to use quadratic elements. A possible choice is to consider the endpoints $\xi_1 = x_j$, $\xi_3 = x_{j+1}$ and the midpoint $\xi_2 = (x_j + x_{j+1})/2$. The 3 form functions are

$$N_1(x) = \frac{(x-\xi_2)(x-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} \tag{45}$$

$$N_2(x) = \frac{(x-\xi_3)(x-\xi_1)}{(\xi_2-\xi_3)(\xi_2-\xi_1)} \tag{46}$$

$$N_3(x) = \frac{(x-\xi_3)(x-\xi_1)}{(\xi_2-\xi_3)(\xi_2-\xi_1)} \tag{47}$$

The local approximation of the solution on E_j is therefore

$$\tilde{Q}_j(x,t) = \sum_{k=1}^{3} Q_k(t)N_k(x) \tag{48}$$

We must now establish a weighted residual procedure. There is no immediate variational approach so we turn to a Galerkin procedure and consider the equations

$$(L\tilde{Q}_j(x,t), N_k(x)) = 0 \tag{49}$$

with

$$L = \partial_t + (x+t)\partial_x \tag{50}$$

and

$$(f,g) = \int f(x)g(x)dx \tag{51}$$

The integration leads to a linear system of ODE's of the form

$$\frac{d}{dt}Q(t) + KQ(t) = 0 \tag{52}$$

with K the system stiffness matrix. The ODE system may be solved using standard methods, e.g. Runge-Kutta.