Homework Assignment
Numerical solution of partial differential equations, I (Course 221)
Handed out: Tuesday, August 26, 2003, Due: Tuesday, September 9, 2003

1 Common PDE's

1.1 PDE's in the sciences

Choose one of your favorite science courses that uses mathematical statements of natural or social laws. Pick 3 of these laws. Keep in mind the discussion in class that tracking money in a bank can be as simply expressed as a single addition or as complicated as an integral equation.

1. Determine whether the laws you've picked correspond to integral or local formulations.
2. For laws you determined are local formulations explicitly identify q, f, σ in

$$\frac{\partial q}{\partial t} + \nabla \cdot \mathsf{f} = \sigma \ . \tag{1}$$

3. For laws you determined are integral formulations explicitly identify $q, \mathsf{f}, \sigma, B, \mathsf{n}$ in

$$\int_B q(\mathsf{x}, t+\Delta t) d\mathsf{x} - \int_B q(\mathsf{x}, t) d\mathsf{x} = \tag{2}$$

$$\int_t^{t+\Delta t} \int_{\partial B} \mathsf{f}(\mathsf{x}, \tau) \cdot \mathsf{n}(\mathsf{x}) \ d\mathsf{x} \ d\tau + \int_t^{t+\Delta t} \int_B \sigma(\mathsf{x}, \tau) d\mathsf{x} \ d\tau. \tag{3}$$

1.2 Finite difference convergence

The textbook definition of a derivative is

$$q'(t) = \lim_{h \to 0} \frac{q(t+h) - q(t)}{h}, \tag{4}$$

Program this formula on a computer (Fortran, Matlab, Mathematica) and verify convergence behavior by computing the first derivative of $q(t) = \cos\left(t^{\sin t}\right)$ at $t_0 = 2$ for $h = 10^{-p}$, $p = 1, 2, \ldots, 10$. Evaluate the exact analytical derivative. Organize your results as a series of $\lg - \lg$ plots of the relative error versus h for each of the formulas. Explain your observations.

1.3 What's that method?

At this stage of your career you should have been exposed to a few numerical methods for solving ODE's and/or PDE's. Choose one of these methods and identify the elements that would cast that method in the weighted residual framework, i.e. identify:

1. the function space from which we construct approximations is chosen along with a subset of a basis of this space $\{l_1, l_2, \ldots, l_N\}$;
2. the set of weight functions $\{w_1, w_2, \ldots, w_N\}$;
3. the discretization of the domain $\{\omega_1, \omega_2, \ldots, \omega_M\}$.

1.4 Approximate by drawing

Draw the vector field of the ODE

$$q'(t) = \cos t + q \tag{5}$$

in the $[-1, 1] \times [-1, 1]$ square of the (t, q) plane. Use a reasonable sampling, say 10 points along each direction. Then solve the ODE for the initial condition $q(0) = 0$ by drawing a curve that passes through $(t, q) = (0, 0)$ and is tangent to the vectors previously drawn. Determine what $q(t = 1)$ is by reading the distance from your curve to the t axis. Now solve the ODE analytically to find an exact value for $q(1)$. What is the relative error of your "drawing" solution? You should compute

$$\varepsilon = \left| \frac{q_{exact}(1) - q_{drawing}(1)}{q_{exact}(1)} \right| . \tag{6}$$

From the value of ε obtained how many significant digits from your drawing solution are exact?

1.5 Bonus - Computer arithmetic

Addition and multiplication on the reals are commutative and associative. They each have an identity element which is unique. Do you think this holds for computer arithmetic? Explain your answer. Try to compute the value of the harmonic series

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \tag{7}$$

for $n = 10^k$, $k = 1, 2, \ldots, 6$ in multiple ways. First, summing left to right, then right to left and finally inwards outwards. Do you obtain the same answers? Explain your observations.