Homework Assignment
Numerical solution of partial differential equations, I (Course 221)
Handed out: Tuesday, September 30, 2003, Due: Tuesday, October 14, 2003

3 Diffusion Equation Numerical Experiments

3.1 Analytical solution on an infinite domain

Consider the IVP

$$\begin{gathered} q_t = q_{xx} \\ q(x,0) = \eta(x) = \cos(\pi x) + \sin(\pi x) \end{gathered} \tag{1}$$

for x real. Solve this problem using Fourier transform techniques.

3.2 Analytical solution on a finite domain

Consider the IVBP

$$\begin{gathered} q_t = q_{xx} \\ q(x,0) = \eta(x) = \cos(\pi x) + \sin(\pi x) \\ q(x=0,t) = g_0(t) = 1 \\ q(x=1,t) = g_1(t) = -1 \end{gathered} \tag{2}$$

This problem models the heating in time of a bar initially at temperature $\eta(x)$ which has its ends at constant fluxes $g_0(t) = g_1(t) = \pi$. Compute the analytical solution to this problem using separation of variables. Compare with the solution from the previous problem.

3.3 Experimenting with FTCS convergence

Now solve the above finite-domain problem using the FTCS algorithm to find the temperature distribution for $t \in [0, 0.5]$. Study the convergence of the method by:

1. Taking succesively smaller step sizes (k,h) in the FTCS discretization. Try three sequences:

   (a) $h = 2^{-p}$, $k = (p+1)h^2/(2p)$, $p = 2,3,4,5,6$;

   (b) $h = 2^{-p}$, $k = h^2/2$, $p = 2,3,4,5,6$;

   (c) $h = 2^{-p}$, $k = (p-1)h^2/(2p)$, $p = 2,3,4,5,6$;

2. Computing the relative error between the numerical approximation and the analytical solution

$$\varepsilon(t) = \frac{\|U_{num}(t) - u_{exact}(t)\|}{\|u_{exact}(t)\|} . \tag{3}$$

Use the 1−norm

$$\|U_{num}\| = \sum_{j=1}^{m} |U_j| \tag{4}$$

in the above computation of the relative error ($h = 1/(m+1)$).

Plot the relative errors at $t_n = n/10$, $n = 1, 2, \ldots, 5$ as a function of h in log-log coordinates. Does the observed convergence behavior verify the theoretical predictions for the FTCS method? Comment on the difference in results for the three sequences of (k, h) specified above.

3.4 Probing Crank-Nicolson convergence

Now solve the IVBP from problem 3 using the Crank-Nicolson scheme. Use a tridiagonal solver inside of your Crank-Nicolson time step. Study the convergence of the Crank-Nicolson method by intelligently reducing the step sizes (k, h). Again, plot your results as log-log graphs of the relative error versus h at different times. Do you obtain the predicted theoretical behavior? How do the FTCS and the Crank-Nicolson approaches compare from the viewpoint of practicality?

3.5 Bonus: 2D ADI and boundary conditions

Consider the 2D problem

$$\begin{aligned} q_t &= q_{xx} + q_{yy} \\ q(x, y, 0) &= \sin(\pi x y) \end{aligned} \tag{5}$$

with zero conditions on $q(x, y, t)$ on the boundary of the unit square $[0, 1] \times [0, 1]$. How do you generate boundary conditions for the intermediate stage value Q^* when applying the ADI (alternating direction implicit) algorithm to this problem?